Editor's comment: Roland Stuff died in March 2009 before this manuscript could go through the review process. Consequently, the paper was read for accuracy by an Associate Editor and is published to honor Stuff's enthusiastic presentation in Lüsse.

# Non-Viscous Vortex Generation due to Buoyancy, an Example of Application of Compulsive Forces in Fluids


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## Abstract

In the application of the Euler momentum equation to the flow of a two-dimensional incompressible vortex pair, the density of the accompanying fluid of which is greater than the one of the ambient fluid, the generation of a pressure discontinuity, i. e. the generation of a shock, at the density discontinuity is predicted. However, even weak shock waves are propagating at the infinite speed of sound in an ideal incompressible fluid, which is a good approximation for low subsonic flows of air. In order to avoid the pressure discontinuity, the condition of pressure continuity is introduced as a compulsive condition. If this condition is not fulfilled, which up to now is the case in relevant papers, the compulsive condition of pressure continuity has to be introduced. This results in a supplementary compulsive force, which in the present case is non-conservative, and consequently leads to the generation of circulation in the barotropic, non-viscous fluid accompanying the vortex pair, for which Bjerknes theorem of baroclinic generation of circulation does not hold. It may be applied to the velocity jump across the density discontinuity, but without the contribution from the compulsive forces does not satisfy the condition of pressure continuity and the cinematic condition at the stagnation points. Thus, another exception to Kelvin's theorem of the constancy of circulation is found. The theory presented is applied to the interference between an aircraft wake and atmospheric instability. In the case of a latent atmospheric instability the aircraft wake may trigger off instability.


## Nomenclature

| | |
|---|---|
| $g$ | gravity acceleration |
| p | static pressure, (pa) |
| $p_0$ | static pressure due to stratification, Eq. (2) |
| R | radius of streamline curvature |
| $\pm R_0$ | distance of vortex canters from the vertical axis |
| s | distance measured along streamline from lower stagnation point |
| v | velocity of accompanying flow in non-inertial reference frame |
| $v_A$ | velocity of ambient flow in non-inertial reference frame |
| W | velocity of the non-inertial reference frame |
| z,y,t | rectangular coordinates and time in non-inertial reference frame |
| Z,Y,T | rectangular coordinates and time in inertial reference frame |
| $\alpha$. | factor of acceleration of the apparent mass |
| $\alpha_A$ | factor of apparent mass at separating streamline |
| $\Gamma$ | circulation |
| $\gamma$ | intensity of vortex sheet, Eq. (26) |
| $\varphi$ | pressure term, Eq. (20) |
| $\rho$ | density of accompanying fluid |
| $\rho_A$ | density of ambient fluid |

*Subscripts*

| | |
|---|---|
| A | ambient flow |
| b | bottom of ambient fluid |
| i | inside accompanying fluid |
| s | vortex sheet |

## Introduction

Non-viscous ideal flows generally are described by the Euler momentum equations. The forces on the right hand side of those equations are conservative, if the flow is barotropic, meaning, that density and pressure gradients are parallel. Conservative means, that the sum of the kinetic and potential energy, i. e. the total pressure of a fluid element remains unchanged. Consequently the curl over these forces is zero as has been shown by Lord Kelvin[1] in his theorem of the constancy of circulation in non-viscous flows. For baroclinic flows, in which pressure and density gradients are not parallel, Bjerknes[2] found his theorem of the generation of circulation, a first exception to Lord Kelvin's theorem.

A non-conservative force is defined such[3] that the work done by this force per unit time equals the rate of change of the sum of its kinetic and potential energy. An example are frictional forces, which are dissipative and, thus, subtracting kinetic and, or, potential energy from the fluid element. In addition, forces imparting energy to the element of fluid, for example, by work of displacement also are non-conservative. Those are dealt with in the present paper as compulsive forces, which are well known for solid bodies, but much less so in fluid mechanics. No example of application in fluid mechanics could be found. The Gaussian principle of least compulsion is mentioned briefly[4]. The condition of pressure continuity must be introduced as a compulsive condition in incompressible flows, if this condition is not already fulfilled by describing it using the Euler momentum equation[5]. This is quite obvious, since in incompressible flows even weak pressure disturbances propagate at the infinite speed of sound. Shock waves cannot exist; the pressure is the same in all directions, i.e. isotropic. Compulsive forces are determined from the resulting accelerations such, that pressure continuity is achieved. They are not furnished a priori, are among the unknowns of the problem, reflect the influence of the entire flow on the local element of fluid and must be obtained from the solution we seek. In the present case, compulsive forces are time dependent. The curl over these forces is non-zero leading to another theorem of the generation of circulation in barotropic flows, a second exception to Lord Kelvin's theorem[1].

In order to reduce the problem to the relevant parameters, a two-dimensional, incompressible, non-viscous, buoyant vortex pair is chosen, the accompanying fluid of which is heavier than the ambient fluid. The description of the flow is completely analytical. Due to the incompressibility assumption, there only is a density discontinuity at the streamline separating the accompanying from the ambient fluid. The Bjerknes theorem is applied to the density discontinuity, but leads to a pressure discontinuity at the separating streamline. This way, the same problem already has been treated[6]. Based on the paper[6], a numerical code for the NASA aircraft wake programme was developed[7, 8]. In the present paper, the pressure discontinuity is avoided by the introduction of compulsive forces. The circulation generated barotropically represents the major fraction of the total circulation.

In the following section, the ambient flow is described using the Euler momentum equations. Adaptation of the accompanying fluid to the increasing total pressure by means of compulsive forces is discussed in the next section. The inevitability of pressure discontinuity by being restricted to the Bjerknes theorem is shown in the subsequent section. For the flow of the accompanying fluid in the non-inertial reference frame compulsive forces are introduced, and, since they are non-conservative, a theorem of the generation of circulation in non-viscous, barotropic flows is derived. The theory is applied to an aircraft's vortex wake triggering off an atmospheric instability.

## Ambient flow, Euler equation

A non-viscous, incompressible vortex pair moves downward at a constant acceleration, since the fluid containing the pair has a higher density, ρ, than the one, $\rho_A$, of the ambient fluid. The non-inertial coordinate system, y, z, and t, is attached to its origin at the centre of the vortex pair and moves downward with it at the velocity, W; see also Fig, 1. The lateral distances of the two vortices from the vertical axis are $\pm R_0$. The origin of the inertial coordinate system, Y, Z, and T, is attached to the bottom of the ambient fluid. $Z_0$ is the altitude of the vortex pair at some initial time. The relationship between the two systems is given by,

$$Z - Z_0 = z + \int W dt\,; \quad Y = y; \quad T = t,$$
$$z = Z - Z_0 - \int W dT\,; \quad y = Y; \; t = T. \tag{1}$$

The static pressure, $p_0$, decreases with altitude,

$$p_0 = p_b - \rho_A g Z, \tag{2}$$

where $p_b$ is the static pressure at the bottom, i. e. the ground, and g the gravity acceleration. The Euler momentum equation of the ambient flow reads in the non-inertial reference frame,

$$\rho_A \frac{dv_A}{dt} = -\nabla p - \rho_A g - \rho_A \frac{dW}{dt}, \tag{3}$$

where $v_A$ is the velocity as seen from the non-inertial reference frame. The above equation is rewritten by means of Eq. (2),

$$\rho_A \frac{dv_A}{dt} = -\nabla(p - p_O) - \rho_A \frac{dW}{dt}. \tag{4}$$

The streamline component of Eq. (4) is,

$$\rho_A\left(\frac{\delta v_A}{\delta t} + v_A \frac{\delta v_A}{\delta s}\right) = -\frac{\delta}{\delta s}(p - p_0) - \rho_A\left(\frac{dW}{dt} \times \vec{n}\right) \times \vec{n} \tag{5}$$

where s is the distance covered from the lower stagnation point, see Fig. 2, and $\vec{n}$ is the unit vector in the streamline normal direction. The streamline normal component of Eq. (4) reads,

$$\rho_A \frac{v_A^2}{R} = -\frac{\delta}{\delta n}(p - p_0) - \rho_A\left(\frac{dW}{dt} \times \vec{t}\right) \times \vec{t} \tag{6}$$

where R is the radius of streamline curvature and $\vec{t}$ the unit vector in the streamline direction.

Although the flow is unsteady, path lines and streamlines are identical in the non-inertial reference frame and the same as those in the corresponding steady flow; see also the referenced textbooks[9, 10]. The velocity potential and stream function are given by the Cauchy-Riemann differential equations of irrotationality and continuity,

$$\Phi_A = 2R_0 W\left\{\frac{-z}{2R_0} + \Theta_2 - \Theta_1\right\},$$

$$\Psi_A = 2R_0 W\left\{\frac{y}{2R_0} + \ln\frac{r_1}{r_2}\right\}. \qquad (7)$$

Only the velocity, W, of the vortex pair as a whole is a function of time. The angles $\Theta_1$ and $\Theta_2$ are given by,

$$\Theta_1 = \tan^{-1}\left\{\frac{(y-R_0)}{z}\right\}, \Theta_2 = \tan\left\{\frac{(y+R_0)}{z}\right\},$$

whereas the distances of the element under consideration from the vortex centres are given by,

$$r_1 = (y-R_0)^2 + z^2,\ r_2 = (y+R_0)^2 + z^2. \qquad (8)$$

Evaluating the velocity potential, Eq. (7), the velocity along the separating streamline is found to be,

$$v_A = -(1+\alpha)\left(W \times \vec{n}\right) \times \vec{n}. \qquad (9)$$

where α is the factor of the apparent mass. The equation says, that the velocity at the dividing streamline is given by the projection of the maximum velocity, (1+α) W on to the streamline. This is called Munk's rule[11], which applies to the acceleration as well,

$$\frac{\delta v_A}{\delta t} = -(1+\alpha)\left(\frac{dW}{dt} \times \vec{n}\right) \times \vec{n}. \qquad (10)$$

Eqs. (9) and (10) can be applied to any continuously shaped two dimensional body, which in the present case happens to have the shape of the separating streamline of a two dimensional vortex pair. Integration of Eq. (5) over the streamline yields the Bernoulli equation extended by the unsteady term,

$$\left\{(p-p_0) - \frac{\rho_A}{2}W^2 - \alpha\rho_A\frac{dW}{dt}z\right\} + \frac{\rho_A}{2}v_A^2 = 0 \qquad (11)$$

Already at this stage it can be seen, that the sum of potential and kinetic energy is a function of time, since the stagnation pressure increases. The factor of the apparent mass can be given as a function of the angles, and the vertical coordinate,

$$\alpha = \frac{2R_9(\Theta_1 - \Theta_2)}{z}. \qquad (12)$$

With this, Eqs. (9) and (10) hold throughout the ambient fluid. Evaluating Eq. (12) for the lower stagnation point, the factor of the apparent mass is found to be,

$$\alpha_A = \frac{2\pi}{3\sqrt{3}}. \qquad (13)$$

This is the ratio of the lateral axis of the vortex pair divided by its longitudinal axis.

However, one has to have in mind that the compulsive forces are obtained from the solution we seek. Therefore, the above solution for the ambient flow, Eq. (7), has to be checked again after the compulsive forces and their effects are known.

## Accompanying flow, Euler momentum equation, compulsive work of displacement

The Euler momentum equation of the accompanying flow in the non-inertial reference frame reads,

$$\rho\frac{dv}{dt} = -\nabla p - \rho g - \rho\frac{dW}{dt}, \qquad (14)$$

where $\rho$ is the density of the accompanying fluid, v the velocity as seen from the non-inertial reference frame. With Eq. (2) the above one is rewritten to give,

$$\rho\frac{dv}{dt} = -\nabla(p-p_0) - (\rho-\rho_A)g - \rho\frac{dW}{dt} \qquad (15)$$

where $(\rho-\rho_A)g$ is the excess gravity force driving the accelerated flow, since the accompanying fluid is heavier than the ambient fluid, $\rho > \rho_A$. In order to accelerate the apparent mass, a vertical pressure gradient $\alpha_A\rho_A\frac{dW}{dt}z$, see also Fig. 3, is superimposed on the pressure distribution of the quasi-steady flow. This term is added to the pressure term and subtracted from the excess gravity force, giving instead of Eq. (15),

$$\rho\frac{dv}{dt} = -\nabla\left\{(p-p_0) - \alpha_A\rho_A\frac{dW}{dt}z\right\} + \left\{-(\rho-\rho_A)g - \rho\frac{dW}{dt} - \alpha_A\rho_A\frac{dW}{dt}\right\} \qquad (16)$$

Comparison with Eq. (11) for the ambient flow shows that the time dependent term $\rho_A/2\, W^2$ is missing in Eq. (16). It represents the increasing total pressure of the ambient flow, and the accompanying fluid has to cope with it by providing this pressure increase through compulsive work of displacement,

$$\left(\rho_A \frac{dW}{dt}\right)\cdot\left[Z - Z_0\right] = \frac{\rho_A}{2} W^2 \Big|_{Z_0}^{Z} \qquad (17)$$

Therefore, the compulsive force of displacement $\rho_A \frac{dW}{dt}$ is added to the Euler momentum equation,

$$\begin{aligned} \rho \frac{dv}{dt} &= -\nabla\left\{\left(p - p_0\right) - \frac{\rho_A}{2} W^2 - \alpha_A \rho_A \frac{dW}{dt} z\right\} \\ &+ \left\{-\left(\rho - \rho_A\right) g - \rho_A \frac{dW}{dt} - \rho \frac{dW}{dt} - \alpha_A \rho_A \frac{dW}{dt}\right\} \end{aligned} \qquad (18)$$

The time rate of change of the work of displacement is equal to the change of total pressure,

$$\left(\rho_A \frac{dW}{dt}\right)\cdot W = \frac{\delta}{\delta t}\left\{\left(p - p_0\right) + \frac{\rho}{2} v^2\right\} \qquad (19)$$

The terms $\alpha_A \rho_A \frac{dW}{dt} z$ and $\int_0^s \frac{\delta v}{\delta t} d\bar{s}$ are not functions of time. The compulsive force of displacement $\rho_A \frac{dW}{dt}$ is non-conservative, since the sum of the potential and kinetic energy[3] increases. In a piston cylinder device the work of compression of an incompressible fluid is zero. However, the accompanying fluid, whose excess gravity force drives the entire process, is not bounded by solid walls. For convenience the pressure term in the above equation is abbreviated by,

$$\varphi = \left(p - p_0\right) - \frac{\rho_A}{2} W^2 - \alpha_A \rho_A \frac{dW}{dt} z \qquad (20)$$

Integrating Eq. (18) over the streamline yields the Bernoulli equation extended by the unsteady term,

$$\begin{aligned} &\varphi + \frac{\rho}{2} v^2 + \rho \int_0^s \frac{\delta v}{\delta t} ds \\ &+ \left\{\begin{aligned} &\left(\rho - \rho_A\right) z + \rho_A \frac{dW}{dt} z \\ &+ \rho \frac{dW}{dt} z + \alpha_A \rho_A \frac{dW}{dt} z \end{aligned}\right\} = 0 \end{aligned} \qquad (21)$$

Regarding the accompanying fluid as a whole, and taking into account that the pressure term $\varphi$ is balanced by the pressure of the ambient flow, see also Eq. (11), we have,

$$\left\{\begin{aligned} &(\rho - \rho_A) g + \rho_A \frac{dW}{dt} \\ &+ \rho \frac{dW}{dt} + \alpha_A \rho_A \frac{dW}{dt} \end{aligned}\right\} = 0. \qquad (22)$$

This equation also can be obtained from the conservation law of energy applied to the entire accompanying fluid,

$$\left\{\begin{aligned} &(\rho - \rho_A) g \left(Z - Z_0\right) + \frac{\rho_A}{2} W^2 \\ &+ \frac{\rho}{2} W^2 + \alpha_A \frac{\rho_A}{2} W^2 \end{aligned}\right\} = const. \qquad (23)$$

From left to right the terms respectively represent the excess gravity potential, the stagnation pressure, the kinetic energy of the accompanying fluid and the kinetic energy of the apparent mass. The streamline component of the momentum equation, Eq. (18), can be simplified by using Eq. (22),

$$\rho \frac{\delta v}{\delta t} + \rho v \frac{\delta v}{\delta s} = -\frac{\delta \varphi}{\delta s}, \qquad (24)$$

This expression has to be compared with the momentum equation of the ambient flow,

$$\rho_A v_A \frac{\delta v_A}{\delta s} = -\frac{\delta \varphi}{\delta s} \qquad (25)$$

Assuming that initially the dynamic pressure is the same on both sides of the separating streamline, there is no force term available for the local acceleration of the accompanying flow. As a result, the above equations predict the forming of a pressure jump across the separating streamline during the course of the motion, since according to Eq. (10) the dynamic pressure increases in the ambient flow, but does not do so in the accompanying fluid. Saffman[6], himself states that the condition of pressure continuity is not fulfilled, but regards this as a higher order mistake. However, even weak disturbances propagate at the infinite speed of sound in an incompressible fluid. Also, applying Bjerknes' theorem to the vortex sheet at the density discontinuity does not solve the discrepancy as shown below.

## Flow field induced by the vortex sheet, Bjerknes theorem

The instantaneous local intensity of the vortex sheet is given by the velocity jump across the separating streamline,

$$\gamma = v_A - v\,. \tag{26}$$

The flow field induced by the vortex sheet is obtained from integrating over the entire vortex sheet. One then obtains for the accompanying flow a downward velocity,

$$\Delta W_S = \frac{(v_A - v)}{v_A} W \tag{27}$$

The velocity induced is the same throughout the accompanying fluid, because in the integration over the vortex sheet the vector from the fluid element to the vortex sheet turns by 360 degrees. The subscript s denotes vortex sheet. For the ambient flow we obtain a different velocity field, which also would be obtained from the two singularities of a vortex pair, because the vector from the fluid element to the vortex sheet does not turn by 360 degrees, instead oscillates and returns to its original angle. The details of the integration may be taken from a textbook[12], which uses a complex potential. The effect of the baroclinic circulation on the ambient flow may be alternatively obtained by an adjoint potential,

$$\Phi_S = \left(1 - \sqrt{\frac{v}{v_A}}\right)\Phi_A\,. \tag{28}$$

where $\Phi_A$ is taken from Eq. (7). The theorem of Bjerknes[2] predicts a time rate of change of circulation due to the baroclinic torque. Evaluating the acceleration over a closed path, see Fig. 4, the time rate of circulation will be found. $\Gamma_S$ is the circulation around the end of the vortex sheet cut off at s. The local intensity is given by,

$$\gamma = \frac{\delta \Gamma_S}{\delta s} \tag{29}$$

and the time rate of change of $\gamma$ is,

$$\frac{\delta \gamma}{\delta t} = \frac{\delta}{\delta t}(v_A - v) \tag{30}$$

The flow field obtained from the Bjerknes theorem cannot fulfil the condition of pressure continuity, as may become evident from the two flow fields described below.

In the first case it is assumed that in the non-inertial reference frame there is no velocity in the accompanying fluid, v = 0. Then the vortex sheet induces at the two stagnation points the velocity W at which the vortex pair moves downward, see Eq. (27), but there is already a pressure jump across the separating streamline in the initial flow.

The second case has been partly explained in the preceding paragraph. In addition to the formation of a shock, the flow does not fulfil the cinematic condition at the two stagnation points, see also Eq. (27). The flow would be torn apart.

## Compulsive condition and compulsive forces in the accompanying fluid

In order to avoid the formation of a shock, the time rate of change of pressure, which a fluid element of the accompanying flow undergoes during the course of the motion has to be introduced as a compulsive condition. The pressure imposed on a fluid element of the accompanying flow by the ambient flow during the course of the motion has a convective and a local time dependent term. Using the abbreviation of Eq. (20), we have for the inertial system,

$$\frac{d\varphi}{dT} = \nabla \varphi \bullet (W + v) + \frac{\delta \varphi}{\delta T} \tag{31}$$

This can be rewritten for the non-inertial system by using Eq. (1),

$$\frac{d\varphi}{dt} = \nabla \varphi \bullet v + \frac{\delta \varphi}{\delta t}\,. \tag{32}$$

It represents the total time rate of change of $\varphi$ as experienced by an element of fluid moving along a streamline at the velocity v on the inside of the separating streamline. Division of the above equation by the velocity v yields the pressure change of the fluid element while passing a local position s,

$$\frac{1}{v}\frac{d\varphi}{dt} = \nabla \varphi + \frac{1}{v}\frac{\delta \varphi}{\delta t} \tag{33}$$

Thus, the convective term on the right is converted into the pressure gradient, which is well known from the Euler momentum equation. The latter, however, does not take into account the second local term. In the ambient flow, the second local term is a consequence of the acceleration obtained from the Euler momentum equation. But, in the accompanying flow it does not appear, see also Eqs. (24) and (25). This is the reason the Euler momentum equation predicts the generation of a pressure jump across the separating streamline during the course of the motion. For the same reason, the theory of nonhomogeneous fluids is restricted to steady flows[13], which greatly limits application. By rearranging Eq. (19), it is found that,

$$\rho v \frac{\delta v}{\delta t} = -\frac{\delta}{\delta t}\varphi \tag{34}$$

Therefore, in order to maintain pressure continuity, the second term of the right hand side of Eq. (33) is added to the momentum equation, Eq. (24), in the streamline direction as another compulsive force,

$$\rho \frac{\delta v}{\delta t} + \rho v \frac{\delta v}{\delta s} = -\left(\frac{1}{v}\frac{\delta}{\delta t} + \frac{\delta}{\delta s}\right)\varphi \tag{35}$$

The first term on the right hand side is the compulsive force term missing in Eq. (24). From Eqs. (34) and (35) it follows,

$$\rho v \frac{\delta v}{\delta s} = -\frac{\delta}{\delta s}\varphi \tag{36}$$

which implies that the dynamic pressure remains unchanged across the separating streamline,

$$\rho \frac{v^2}{2} = \rho_A \frac{v_A^2}{2} \tag{37}$$

With Eqs. (9), (10), (34) and (37), the velocity of the accompanying fluid is given by,

$$v = \sqrt{\frac{\rho_A}{\rho}} v_A = -\sqrt{\frac{\rho_A}{\rho}}(1+\alpha)(W \times \vec{n}) \times \vec{n} \tag{38}$$

and the acceleration by,

$$\frac{\delta v}{\delta t} = \sqrt{\frac{\rho_A}{\rho}}\frac{\delta v_A}{\delta t} = -\sqrt{\frac{\rho_A}{\rho}}(1+\alpha)\left(\frac{dW}{dt} \times \vec{n}\right) \times \vec{n} \tag{39}$$

The intensity of the density discontinuity now is,

$$\gamma = v_A - \sqrt{\frac{\rho_A}{\rho}} v_A . \tag{40}$$

The flow field induced by the vortex sheet on the accompanying fluid then is,

$$\Delta W_S = \left(1 - \sqrt{\frac{\rho_A}{\rho}}\right) W \tag{41}$$

The baroclinic generation of circulation is with Eqs. (29) and (30),

$$\frac{\delta}{\delta t}\left(\frac{\delta \Gamma}{\delta s}\right) = \frac{\delta v_A}{\delta t}\left(1 - \sqrt{\frac{\rho_A}{\rho}}\right) \tag{42}$$

since the baroclinic circulation only induces a fraction of the necessary velocity W at the stagnation points, see Eq. (41), another mechanism of vortex generation must make up for this deficit.

### Barotropic generation of circulation

The time rate of change of circulation is:

$$\frac{d\Gamma_i}{dt} = \oint \frac{dv}{dt} ds. \tag{43}$$

The subscript i indicates the inside of the accompanying fluid. Using the solution for the ambient flow as a boundary condition, one obtains for the velocity potential of the inside,

$$\Phi_i = \sqrt{\frac{\rho_A}{\rho}} \Phi_A \tag{44}$$

where $\Phi_A$ is given by Eq. (7). But it now applies to the entire accompanying flow. Evaluating Eq. (43) along a closed streamline by using Eq. (39) for the acceleration, the barotropic time rate of circulation finally is,

$$\frac{d\Gamma_i}{dt} = -\sqrt{\frac{\rho_A}{\rho}} 4\pi R_0 \frac{dW}{dt} \tag{45}$$

with $\frac{dW}{dt}$ given from Eq. (22),

$$\frac{dW}{dt} = \frac{-(\rho - \rho_A)g}{\rho_A + \rho + \alpha_A \rho_A} . \tag{46}$$

Eqs. (45) and (46) describe the barotropic generation of circulation due to buoyancy in an ideal non-viscous fluid. It should be noted that the density is constant throughout the accompanying fluid, which renders the accompanying flow barotropic. At the two stagnation points the contribution of the barotropic circulation to the velocity is,

$$\Delta W_i = \sqrt{\frac{\rho_A}{\rho}} W \qquad (47)$$

With the contribution from the baroclinic circulation, Eq. (41) superimposed on the flow field of the barotropic circulation, we have for the velocity at the stagnation points,

$$\Delta W_S + \Delta W_i = W , \qquad (48)$$

which satisfies the cinematic condition. The solution for the ambient flow, Eq. (7), still holds, if the velocity W is replaced by the total circulation.

$$W = \frac{\Gamma_{total}}{4\pi R_0} = \frac{\Gamma_i}{4\pi R_0} + \frac{\Gamma_S}{4\pi R_0} \qquad (49)$$

Since the square root ratio of the densities is near to 1, it is evident that the major part of the flow emanates from the circulation of the barotropic generation.

## Vortex pair in a stratified atmosphere

A vortex pair moving downward in a stratified atmosphere is isentropically compressed by the increasing static pressure. This causes a supplementary acceleration[14]. The velocity W still may be obtained from Eq. (49). But the distance $R_0$ now is a function of time. Both contributions to the acceleration, the one due to compressibility and the one due to buoyancy may be obtained from a straightforward time differentiation of Eq. (49),

$$\frac{dW}{dt} = \frac{\Gamma_{total}}{4\pi R_0^2} \frac{-dR_0}{dt} + \frac{1}{4\pi R_0} \frac{d\Gamma_{total}}{dt} \qquad (50)$$

The term, $\frac{-dR_0}{dt}$, represents the velocity of the vortex centres towards each other. The acceleration due to buoyancy may be obtained from Eq. (46), if the velocity W is replaced by $\Gamma_{total}/4\pi R_0$. This way, Eq. (50) describes the combined effects of compressibility and buoyancy on the acceleration of the vortex pair.

## Vortex wake of a jumbo jet in an atmosphere with an adverse stratification

A jumbo jet, B-747, is assumed to fly with a weight of 322 tons at 170 knots and 5 km altitude, where the atmospheric state is given by a density of $\rho_A = 0.75423$ kg/m³, and a pressure of $p_0 = 565.5\ hpa$. The temperature lapse rate is slightly over-adiabatic, a = -0.0105° K/m. Then the initial figures for the total circulation, vortex spacing and downward velocity respectively are 822m²/s, 47m, 2.8 m/s. From the initial state at an altitude of 5 km and the temperature lapse rate the atmospheric state at sea level is found to be, $\rho_A = 1.13962$ kg/m³, $p_0 = 1026.2\ hpa$, T = 313.7K. The accompanying air is isentropically compressed from the above mentioned initial density $\rho = 0.75423$ kg/m³ to $\rho = 1.24928$ kg/m³ at sea level. Upon descent it is getting progressively heavier than the ambient atmosphere. The increase of the total circulation is found from the buoyancy equation, Eq. (46), to be $\Gamma_{total} = 3784.4$ m²/s, a 4.6 fold increase of the initial circulation of 822m²/s, which has been created by the lift and lift distribution of the aircraft. Out of the increase of circulation, 94.2% emanates from the barotropic generation, and 5.8% from the baroclinic generation at the vortex sheet. At sea level the descent speed of the vortex pair is W=15.9m/s, a 5.7 fold increase of the initial figure of 2.8m/s, out of which 10.5m/s are due to buoyancy and 2.6m/s due to the isentropic compression upon descent. Certainly, these increases cannot be regarded as small.

## Concluding remarks

The non-viscous, incompressible flow of a two dimensional buoyant vortex pair is solved such that pressure continuity is maintained throughout the flow. For the ambient flow an analytic potential and a stream function are found by using the Euler momentum equations. The circulations of those functions initially are undetermined as long as the compulsive forces acting upon the accompanying flow and the resulting accelerations are unknown, since compulsive forces are not furnished a priori and are part of the solution we seek. The compulsive condition of pressure continuity across the separating streamline is obtained by a compulsive force doing work of displacement at the expense of the excess gravity potential of the heavier accompanying fluid. This renders the compulsive force non-conservative. In addition, accelerations of the flow as seen from the non-inertial reference frame are found.

A theorem for the generation of circulation in non-viscous barotropic flows is obtained from the curl of the non-conservative force, a second exception to Kelvin's[1] theorem of the constancy of circulation. The first exception is the theorem of Bjerknes[2] of the generation of circulation in baroclinic flows.

The concept of compulsive forces may have far reaching consequences, since every flow phenomenon, a viscous as well as a non-viscous one, leaves a footprint in the pressure distribution, which may enter the problem via the compulsive condition of pressure continuity.

The Bjerknes theorem for baroclinic flows is applied to the vortex sheet at the separating streamline and the new theorem for barotrpic flow is applied to the inside the accompanying fluid. For the ambient flow the circulation is given by the sum of the baroclinic and barotropic contributions. This way, the initial solution for the ambient flow is substantiated.

The vortex sheet induces throughout the accompanying flow a constant velocity and the barotropic circulation induces the flow field of a vortex pair, the speeds of which are reduced by the square root ratio of the densities, so that the dynamic pressure is unchanged across the separating streamline. The addition of both fields fulfils the cinematic conditions at the two stagnation points.

The example of a jumbo jet demonstrates that the major part of the circulation of the vortex wake emanates from the barotropic generation. Only a small fraction of circulation is generated by the baroclinic vortex sheet at the separating streamline.

Thus, an aircraft flying in an unstably stratified atmosphere may trigger an instability. This possibility might be considered by wake vortex warning systems. Accident investigation committees might consider the vertical atmospheric profiles when a wake-circulation accident occurred. Additional efforts might be undertaken by the aircraft industry to obtain a fast decay of wake circulation through aerodynamic measures.

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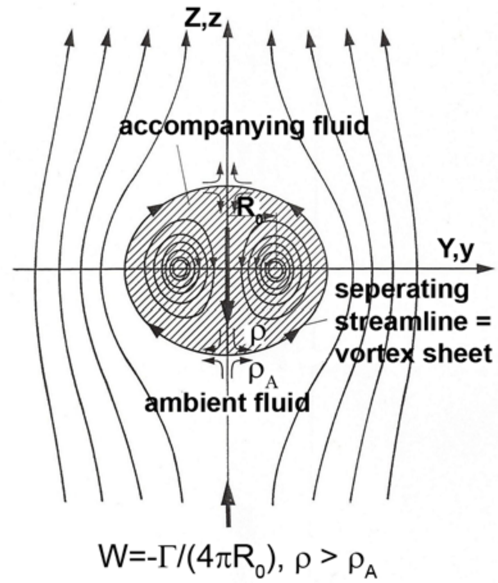


**Figure 1** Inertial and non-inertial reference frame respectively given by Z, Y, T and z, y, t. The streamlines of the two-dimensional vortex pair belong to the non-inertial reference frame. The accompanying fluid has a higher density than the ambient one, $\rho > \rho_A$. Both fluids are assumed to be incompressible. The separating streamline is a density discontinuity. Buoyancy accelerates the vortex pair downward.

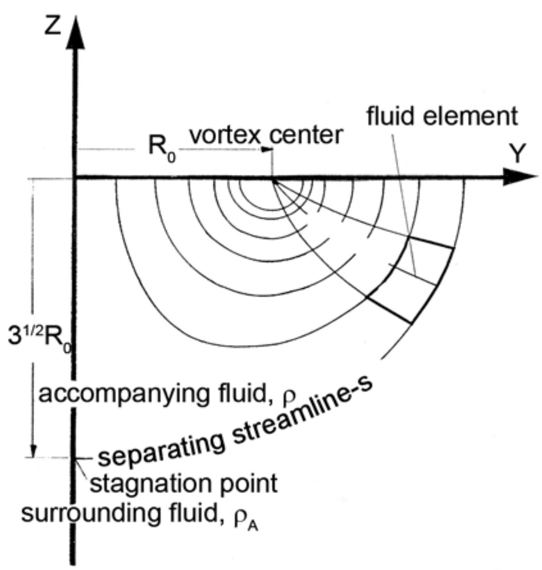


**Figure 2** The coordinate s is measured along the separating streamline, starting at the lower stagnation point. At this point the combined circulation, the baroclinic one of the vortex sheet and the barotropic one from the two single vortices, must induce the velocity W, at which the vortex pair moves downward. Otherwise the flow would be torn apart, which is not possible.

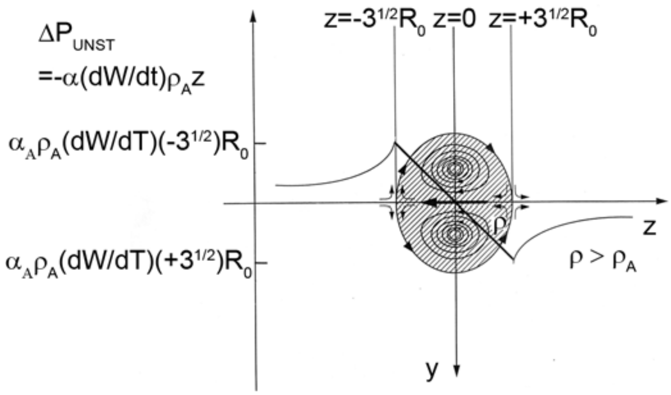


**Figure 3** Pressure distribution due to the acceleration of the apparent mass, which is superimposed on that one due to the quasi-steady flow. In order to plot the pressure on a vertical line, the z-coordinate is turned into the horizontal. Below the vortex pair (left hand side) the apparent mass is accelerated by overpressure above (right hand side) by suction.

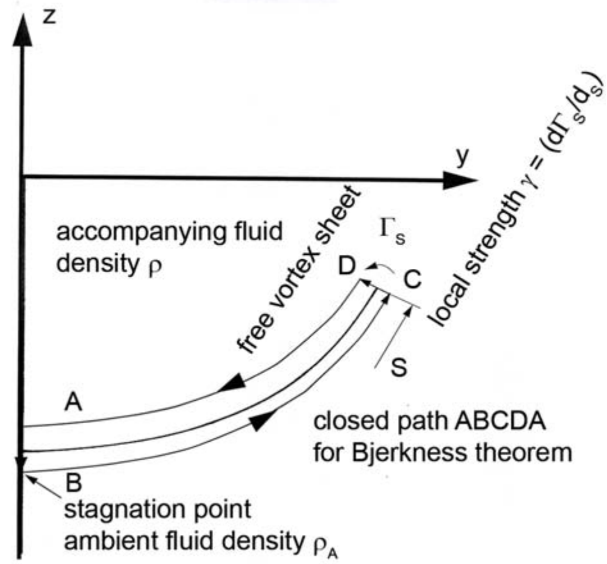


**Figure 4** Closed path ABCD for evaluation of Bjerknes' theorem applied to the vortex sheet along the separating streamline. AB and CD are at constant pressure and do not contribute. The contributions come from BC at density $\rho$ and DA at density $\rho_A$. $\Gamma_S$ is the circulation around the end of the free vortex sheet cut off at s.