HEAVILY LOADED, GLUED JOINTS

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Abstract

The simplified theory of glued joints is applied to joints where not only the thickness of the glued elements but also the thickness of the glue layer itself may be variable. An ordinary differential equation is derived for the shear stress in the glue layer of a two-dimensional problem. The conventional scarf joint is one of the solutions. Constant thickness of the glued elements and the glue layer (lap joint) show very high stress concentrations at the ends of the glue layer. If a scarf joint is not possible, a variable-thickness glue layer can reduce these stress concentrations.

1. Introduction

A glued joint as shown in Figure 1 under a longitudinal force F is considered. The conventional simplified theory assumes [l]

- the problem is two-dimensional, the force F is a force per unit length perpendicular to the plane of the sketch,
- the two elements that are glued together undergo only longitudinal deformations, and
- in the glue layer between the two elements, only the shear deformation is considered.

These assumptions have been validated by means of better studies, for example, [2]. The errors are small, the simplified theory shows the most significant features quite well. The scarf joint is an optimal solution, whereas the "lap joint" with constant thickness of the glued elements and the glue layer yields a high stress concentration at the ends of the glue layer. It is thus attractive to use a scarf joint whenever possible. This may not always be possible, however. An example is a cambered spar consisting of two caps and two shear webs. Under a bending moment, the caps have a strong tendency to increase (or decrease) their spacing. The webs must then carry high longitudinal forces perpendicular to the caps. In this case, the thickness t_1 of the cap (perpendicular to the web) is much larger than the thickness of the web and a scarf joint is impossible. For this reason, the present paper considers not only variable thicknesses $t_1(x)$ and $t_2(x)$ of the glued elements but also a variable thickness $t_g(x)$ of the glue layer. The simplified theory is extended to this variable $t_g(x)$ in the next section. The lap joint and the scarf joint are simple special cases of the more general theory.

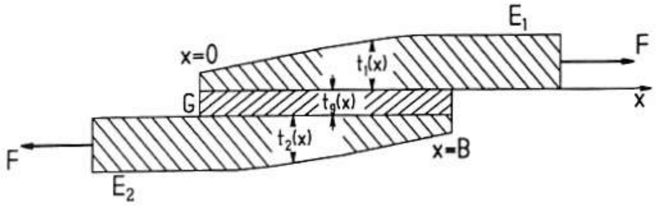

Figure 1. Sketch of the glued joint.

2. The Generalized, Simplified Theory

The differential equation for the shear stress in the glue layer is derived by using Figure 2. This figure is enlarged in the vertical direction. In particular, the thickness t_g of the glue layer is smaller in the reality.

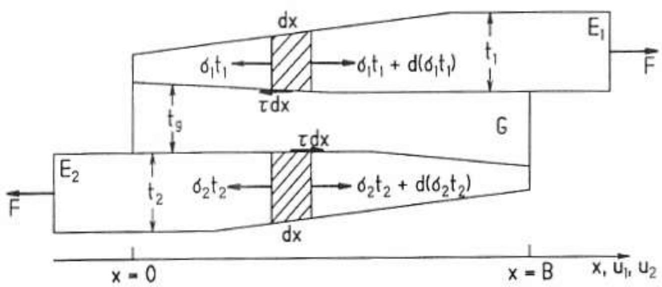

Figure 2. Sketch of the forces in a glued joint

Two small sections of the glued elements, shown shaded in the sketch, are considered. According to the assumptions previously discussed, only normal stresses act on the vertical boundaries of these sections; only a shear stress, on the boundary with the glue layer. The forces on this section must be in equilibrium. The forces are the stresses multiplied with the areas on which the stresses act. The extension in the direction perpendicular to the plane of the sketch is the same for all areas on which stresses act. Therefore, equilibrium of the forces per unit length is established. This yields, for the upper and the lower sections

$$\tau\, dx = d(\sigma_1 t_1), \qquad -\tau\, dx = d(\sigma_2 t_2), \qquad (1)$$

or

$$\tau = \frac{d(\sigma_1 t_1)}{dx}, \qquad -\tau = \frac{d(\sigma_2 t_2)}{dx}, \qquad (2)$$

where σ_1 and σ_2 are the normal stresses in the glued elements, and τ is the shear stress in the glue layer.

To establish the equation for the deformation, it is assumed that the elastic moduli E_1 and E_2 of the glued elements and the shear modulus G of the glue layer are constant. The displacement of the upper and lower elements are designated u_1 and u_2. Although these displacements are not shown in Figure 2, it is easy to realize that the longitudinal deformation of the upper element is $\varepsilon_{x1}=du_1/dx$. The same holds for the lower element with subscript 2. (Note that constant displacement would not cause a deformation.) Hooke's law then yields for both elements

$$\frac{du_1}{dx} = \epsilon_{x_1} = \frac{\sigma_1}{E_1}, \qquad \frac{du_2}{dx} = \epsilon_{x_2} = \frac{\sigma_2}{E_2}. \qquad (3)$$

The shear stress in the glue layer is caused by the different displacements of the upper and lower elements. Thus,

$$\frac{\tau}{G} = \frac{u_1 - u_2}{t_g}. \qquad (4)$$

Equations (2) to (4) are adequate for the solution of the problem. The remainder is simple mathematics. Insertion of σ_1 and σ_2 from (3) into (2) yields

$$\tau = \frac{d}{dx}\left(E_1 t_1 \frac{du_1}{dx}\right), \qquad -\tau = \frac{d}{dx}\left(E_2 t_2 \frac{du_2}{dx}\right). \qquad (5)$$

Because E_1 and E_2 are constant, they can be moved in front of the differentiation. Then τ can be inserted from (4). This yields the following, two coupled ordinary differential equations for and u_1 and u_2

$$\begin{aligned} \frac{d}{dx}\left(t_1 \frac{du_1}{dx}\right) - \frac{G}{E_1 t_g}(u_1 - u_2) &= 0 \\ \frac{d}{dx}\left(t_2 \frac{du_2}{dx}\right) + \frac{G}{E_2 t_g}(u_1 - u_2) &= 0 \end{aligned} \qquad (6)$$

The boundary conditions are

$$\begin{aligned} x = 0: &\quad \sigma_1 t_1 = 0, \quad \sigma_2 t_2 = F \\ x = B: &\quad \sigma_1 t_1 = F, \quad \sigma_2 t_2 = 0 \end{aligned} \qquad (7)$$

These ordinary differential equations for $u_1(x)$ and $u_2(x)$ can be solved for given functions $t_1(x)$, $t_2(x)$, *and* $t_g(x)$. The most important result $\tau(x)$ follows from (4). It is also possible, however, to pre-specify other functions like $\tau(x)$ and $t_g(x)$. The unknown functions are then $t_1(x)$ and $t_2(x)$. The third option, which is the subject of the present paper, is to specify $t_1(x)$, $t_2(x)$, *and* $\tau(x)$. The unknown function is then $tg(x)$, the thickness of the glue layer.

3. Scarf Joints

The simplified theory of glued joints can be used to determine if scarf joints are really optimal solutions. Scarf joints are optimal if $\tau(x)$ = *constant* for t_g = *constant* and linear $t_1(x)$ and $t_2(x)$. Under these conditions, it follows from (4) that

$$u_1 - u_2 = \frac{t_g \tau}{G} = \text{constant}. \qquad (8)$$

Then (6) can be written for constant $\underline{E}_1$ and E_2 together with (3), as

$$\frac{d}{dx}(t_1\sigma_1) = \tau, \qquad \frac{d}{dx}(t_2\sigma_2) = -\tau. \qquad (9)$$

Because τ is constant, (9) yields with the boundary conditions (7)

$$t_1\sigma_1 = \frac{x}{B}F, \qquad t_2\sigma_2 = (1-\frac{x}{B})F. \tag{10}$$

The conventional scarf joint has linear functions for t_1 and t_2. Thus, σ_1 and σ_2 must also be constant. From (8) and (3), it follows that $\sigma_1/E_1 = \sigma_2/E_2$. From (10), it follows that the scarf joint is only possible if

$$t_2(0) = \frac{E_1}{E_2}t_1(B). \tag{11}$$

This is an important condition for scarf joints. If, for example, both glued elements consist of the same material, then the scarf joint is only optimal if the glued elements have the same thickness outside of the glued section.

If condition (11) is not satisfied, an optimal solution with constant τ and t_g is still possible but then the thickness functions are no longer linear.

4. Lap Joints

There are cases in which scarf joints are not possible or too expensive. The simplest case of glued joints is the lap joint, where all three thicknesses t_1, t_2, and t_g are constant. A good lap joint is imagined to have a very thin glue layer. In this case the ordinary differential equations (6) can be simplified to

$$\begin{aligned} \frac{d^2u_1}{dx^2} - \frac{G}{E_1t_1t_g}(u_1-u_2) = 0, \\ \frac{d^2u_2}{dx^2} + \frac{G}{E_2t_2t_g}(u_1-u_2) = 0. \end{aligned} \tag{12}$$

With $v = u_1 - u_2$, it follows that

$$\frac{d^2v}{dx^2} - \alpha^2 v = 0, \qquad \text{where} \qquad \alpha^2 = \frac{G}{t_g}\left(\frac{1}{E_1t_1} + \frac{1}{E_2t_2}\right). \tag{13}$$

The boundary conditions for v follow from (7):

$$\begin{aligned} x = 0: \quad & \sigma_1 = 0, \quad \sigma_2 = \frac{F}{t_2}, \\ x = B: \quad & \sigma_1 = \frac{F}{t_1}, \quad \sigma_2 = 0 \end{aligned}$$

This means, that for v

$$\frac{dv}{dx}(x=0) = -\frac{F}{E_2t_2}, \qquad \frac{dv}{dx}(x=B) = \frac{F}{E_1t_1} \tag{14}$$

The general solution of (13) is

$$v(x) = C_1e^{\alpha x} + C_2e^{-\alpha x}. \tag{15}$$

The integration constants C_1 and C_2 are determined by the boundary conditions. They are

$$\begin{aligned} C_1 &= \frac{F}{2\alpha\sinh(\alpha B)}\left(\frac{1}{E_1t_1} + \frac{e^{-\alpha B}}{E_2t_2}\right), \\ C_2 &= \frac{F}{2\alpha\sinh(\alpha B)}\left(\frac{1}{E_1t_1} + \frac{e^{\alpha B}}{E_2t_2}\right). \end{aligned} \tag{16}$$

The function $v(x)$ according to (15) is a "bath tub" function which increases exponentially towards both ends. The shear stress τ is known because

$$\tau = \frac{G}{t_g}. \tag{17}$$

The maximum values of τ occur at both ends of the glue section. They are

$$\begin{aligned} \tau(0) &= \frac{G\,F}{\alpha t_g\sinh(\alpha B)}\left(\frac{1}{E_1t_1} + \frac{\cosh(\alpha B)}{E_2t_2}\right), \\ \tau(B) &= \frac{G\,F}{\alpha t_g\sinh(\alpha B)}\left(\frac{1}{E_2t_2} + \frac{\cosh(\alpha B)}{E_1t_1}\right). \end{aligned} \tag{18}$$

From

$$\alpha B = \sqrt{\frac{BG}{t_g}\left(\frac{1}{E_1t_1} + \frac{1}{E_2t_2}\right)} \gg 1, \qquad \coth(\alpha B) \approx 1, \tag{19}$$

it follows that

$$\begin{aligned} \tau(0) &\approx \frac{F}{B\sqrt{\frac{t_gt_2E_2}{B^2G}\left(1 + \frac{E_2t_2}{E_1t_1}\right)}}, \\ \tau(B) &\approx \frac{F}{B\sqrt{\frac{t_gt_1E_1}{B^2G}\left(1 + \frac{E_1t_1}{E_2t_2}\right)}}. \end{aligned} \tag{20}$$

The average τ would be F/B. The square root in the denominator of (20) is very small. The large stress concentrations mentioned in section 1 occur at the edges $x = 0$ and $x = B$. The thinner the glue layer, the higher the stress concentrations.

5. Variable Thickness of the Glue Layer

The thickness of the glue layer was found to be a significant feature of the lap joint. The question may be raised if a variable thickness of the glue layer could improve a lap joint, or any other glued joint in which high stress concentrations occur. The fundamental idea is described for the example of the lap joint. It is assumed that the thicknesses of the glued elements are again constant. It is attempted to achieve constant τ by means of variable thickness t_g of the glue layer.

If τ is constant, its value is $\tau = F/B$. From (4), it follows that

$$t_g = \frac{BG}{F}(u_1 - u_2) \tag{21}$$

and the ordinary differential equations (6) are now

$$\begin{aligned} \frac{d^2u_1}{dx^2} &= \frac{F}{E_1t_1B} = \text{constant}, \\ \frac{d^2u_2}{dx^2} &= \frac{F}{E_2t_2B} = \text{constant}. \end{aligned} \tag{22}$$

The subtraction of the second equation from the first one yields, together with (21),

$$\frac{d^2t_g}{dx^2} = G\left(\frac{1}{E_1t_1} + \frac{1}{E_2t_2}\right) = \beta = \text{constant}. \quad (23)$$

The boundary conditions must again be derived from (7), (3), and (21). It follows that

$$\frac{dt_g}{dx}(x=0) = -\frac{GB}{E_2t_2}, \qquad \frac{dt_g}{dx}(x=B) = \frac{GB}{E_1t_1}. \quad (24)$$

The general solution of (22) is

$$t_g(x) = \frac{\beta}{2}x^2 + C_1x + C_2. \quad (25)$$

From the first boundary condition, it follows that

$$C_1 = \frac{dt_g}{dx}(x=0) = -\frac{GB}{E_2t_2}. \quad (26)$$

Insertion of (26) into (25) yields

$$\frac{dt_g}{dx}(x=B) = B\beta + C_1 = \frac{GB}{E_1t_1} \quad (27)$$

The second boundary condition is thus automatically satisfied; C_2 remains undetermined. This result already follows from the fact that the boundary conditions (24) concern only the first derivative of t_g in which C_2 does not appear. This is clear because adding a constant value to t_g means for constant τ, merely adding a constant value to $u_1 - u_2$, which does not influence all equations of the problem. It is most simple to set $C_2 = C^2_1/(2\beta)$ and to write the solution (25) as

$$t_g(x) = \frac{\beta}{2}(x-x_0)^2, \quad \text{with} \quad x_0 = -\frac{C_1}{\beta} = \frac{E_1t_1B}{E_2t_2 + E_1t_1}. \quad (28)$$

The minimum of t_g is set to zero this way. There is indeed a solution with constant τ for the lap joint if variable t_g is allowed. The form of this solution is a parabola. However, the thickness t_g may become unrealistically large. The ratio of the maximum thickness to the half length of the glue layer should be small. It holds, for example, that

$$\frac{t_g(B)}{B-x_0} = \frac{G(B-x_0)}{2}\left(\frac{1}{E_1t_1} + \frac{1}{E_2t_2}\right) \quad (29)$$

This ratio will be a small value only if G/E_i is small, which may be true if composite materials are glued by means of resins. If the glue layer were required to become too thick, the optimal parabola need only to be applied near the edges of the glue joint. This already reduces the stress concentration already very much. In fact, conventional, glued lap joints may be much better if their edges are not very precise. If the glue layer becomes thicker there, the joints may carry much higher loads.

6. Tangentially Loaded Glued Joints

The simplified theory of glued joints can also be applied to joints that carry tangential loads. All formulas of the present paper can be applied to this problem if the elastic moduli E_1 and E_2 are replaced by the shear moduli G_1 and G_2 of the glued elements. In this case the parabolic thickness of the glue layer becomes thicker than for the case with loads perpendicular to the glued section. These results apply, for example, to conventional spars of gliders, where a high shear load in the webs of the spar must be transferred to the caps. This is probably the reason why the permitted stresses in the webs are much smaller than the stresses that their material can carry. It seems to be important to improve these critical points.

7. Perspectives and Future Research

The simplified theory of glued joints yields good approximations for the critical points of joints. The results should be checked by means of better theories, for example, finite-element methods. Such research has begun in Stuttgart. The finite-element method also allows combined normal and tangential loads of the glued joints to be studied. Moreover, experimental verification of the theoretical results are very desirable. Such experiments have not yet been initiated. The problem of variable thickness of the glue layer appears not to be too difficult to resolve. Perhaps preproduced tapes can be used that are glued between the glued elements.

The projected improvements of heavily loaded, glued joints surely justify further research.

References

[1] Volkersen, 0.: Die Schubkraftverteilung in Leim-, Niet- und Bolzenverbindungen, Energie und Technik 1953, pp. 68 - 71, 103- 108, 150- 154.

[2] Benson, N. K.: Influence of Stress Distributions on the Strength of Bonded Joints, unknown report, pp. 191- 205, copy at the author.