Prediction of Lee Wave Factors

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Summary
Wavelength, wave amplitude, and vertical currents in lee waves are functions of vertical profiles of the wind and temperature of the airstream containing the waves and the underlying size and shape of mountain ridges triggering off the waves. The functions are too complicated for routine prediction purposes, but simplifications can be made by considering a real airstream as a simple two- or three-layer model – each layer being characterised by a parameter referred to in this paper as a natural wavelength.
Techniques using a two-layer model are adequate for many airstreams, but can be very inaccurate for certain ranges of values of the parameters involved. Furthermore, the two-layer model is unrealistic in that it assumes both a stable layer and fresh winds near ground level.
This paper describes how the parameters can be scaled to permit the practical use of a more realistic three-layer model and to indicate whether or not a prediction can be made with confidence.

1. Introduction

The principal effects of lee waves on aircraft operations are:

(I) Vertical currents, especially downdraughts in lee of mountains,
(II) Turbulence
(III) Local strengthening or chaotic changes of surface winds under a wave pattern.

Subsidiary effects also include variability of winds aloft, changes in the height of freezing level and wave cloud formation, but in this paper only the three effects listed will be considered.
Authors including Scorer (1949, 1953), Turner (1951), Casswell (1956), Wallington & Portnall (1958) and Lester (1975) have contributed to literature on understanding and predicting lee waves, but no simple technique has been evolved for making confident predictions throughout the range of conditions that are favourable for waves to form. The prediction problem is that, while oversimplified versions of lee wave theory are adequate for simple prediction methods on many occasions, the most favourable conditions for waves are likely to be those in which the wavelengths and amplitudes are likely to be very sensitive to slight changes or incorrect estimations of the airstream parameters which provide those conditions.
This paper does not purport to solve this prediction problem; it aims to make the problem more manageable by discussing the nature of lee wave predictions and indicating the approximate ranges of conditions in which oversimplified models are adequate or doubtful.

2. Factors in the Lee Wave Effects

2.1 "Natural Wavelength"

When discussing wave flow it is often expedient to describe the wind and stability in any layer of an airstream in terms of a parameter, λ, defined by the formula

$$\lambda = 2\pi \left(\frac{g}{U^2 T} \left(\frac{\partial T}{\partial z} + \Gamma \right) - \frac{1}{U} \quad \frac{\partial^2 U}{\partial z^2} \right)^{-\frac{1}{2}} \qquad (1)$$

where T and U denote temperature and wind speed at height ,z,
g is the acceleration due to gravity
and Γ is the appropriate (dry or saturated) adiabatic lapse rate.

The last term in the formula is often small and its omission reduces the equation to

$$\gamma = 2\pi U \sqrt{\frac{T}{g\left(\frac{\partial T}{\partial z} + 1^{1}\right)}} \qquad (2)$$

The significance of the stability factor may be appreciated by considering the buoyancy force on a parcel of air displaced vertically from its equilibrium level in a stable environment. For small displacements the parcel would oscillate up and down about its equilibrium level with a period of oscillation equal to

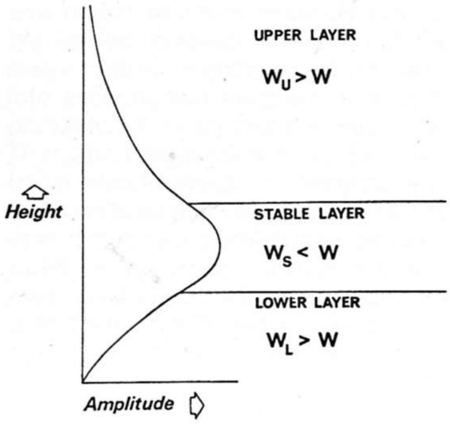
$$2\pi \sqrt{\left(\frac{T}{g}\right) / \left(\frac{\partial T}{\partial z}\right) + \Gamma).}$$

The greater the stability, the shorter the period. Vertical oscillation plus horizontal motion leads to a wave flow whose wavelength is the period multiplied by the wind speed, i. e. λ. Thus λ may be regarded as the natural wavelength of the layer of air in which it is measured (λ = 2π/l, where l is the Scorer parameter used in most of the papers already referred to.)
Natural wavelength usually varies with height in a real airstream. When a lee wave flow exists the lee wavelength is determined entirely by the airstream wind and temperature structure, and usually has one, and occasionally two, values somewhere between the maximum and minimum values of the natural wavelengths in the airstream.

2.2 Variation of Amplitude with Height

In 1949 Scorer devised a method of calculating not only the wavelength on which oscillations could take place in an airstream but also the two-dimensional pattern of streamlines crossing a ridge. Fig. 1 illustrates the nature of the variation of amplitude with height for a "three-layer"» airstream – that is an airstream with three layers distinguishable by their natural wavelengths. In this illustration, the airstream comprises a stable layer (with short natural wavelength) sandwiched between an upper layer with a longer natural wavelength (usually associated with lesser stability and stronger winds) and a lower statically neutral (usually a dry adiabatic layer) with infinite natural wavelength. In this example, Scorer's theory indicates that, provided the values of λ and the

Figure 1
The curve illustrating the typical variation of amplitude with height in a "3-layer airstream".
"W" indicates natural wavelength, with suffix U, S and L for the upper, stable and lower layers. The "W" denotes the resultant natural wavelength for the airstream as a whole.
In any layer where the lee wavelength is greater than the natural wavelength of the layer, the amplitude profile curves in towards the zero amplitude axis; while the profile curves outwards in the other layers. Thus, the maximum amplitude is found in the stable layer.



depths of the layers satisfy certain relationships, the airstream could contain at least one lee wave with a wavelength between the natural wavelengths of the uppermost and middle layers in the illustration. This nature of the amplitude profile is such that maxima can occur only in layers where the natural wavelength is less than the lee wavelength. Thus, in this example, the maximum amplitude will occur in the stable (short natural wavelength) layer.

Although real airstreams usually comprise more than three layers characterised by their natural wavelengths, the three-layer structure often appears to be characteristic of lee waves – as reported by Pilsbury (1955). Even for a three-layer airstream, however, the calculations of lee wavelength and wave amplitude are too laborious for routine practical forecasting without a computing aid; so designers (Casswell (1956) and Lester (1975), for example) of lee wave prediction techniques have used even simpler models. The most tempting simplification is to omit the bottom adiabatic layer, but this means that the base of the stable layer will be at ground level – which, except in very cold climates, is often incompatible with winds over mountain ridges in lee wave conditions usually being about 30 km/hr or more. The technique being advocated in this paper is to use the three-layer model to estimate the maximum lee wave amplitude (and associated lee wave factors), simplification being achieved by assuming the nature of the amplitude profile, rather than by neglecting the bottom adiabatic layer.

2.3 Amplitude Magnitude

Figure 1 illustrates the nature of the amplitude profile, but not the magnitude. The magnitude in lee of a long, two-dimensional cross-wind mountain ridge can be considered as the product of two principal factors:

(I) A Mountain Ridge Factor – which is a product of the two components:
  (a) The height of the ridge above the neighbouring flatter terrain, and
  (b) a ridge width factor determined by a form of resonance between the along-wind width of the ridge and the lee wavelength.

(II) An Airstream Factor – which is a function of the airstream wind and temperature structure only, and which can be considered as the product of two components:
  (a) A wind speed factor, or more specifically, the wind at ground level divided by the wind at the height being considered, and
  (b) a factor which is a function of the airstream's natural wavelength structure only

Hence we can express a lee wave amplitude as

$$A_Z = 2\pi M_H M_W (U_G/U_Z) N_Z \quad (3)$$

where A_Z = Lee wave amplitude at height z
M_H = Height of the mountain ridge triggering off the wave
M_W = The width factor of the ridge triggering off the wave
U_G = Wind speed across the ridge at ground level
U_Z = Wind speed across the ridge at height z
N_Z = Airstream factor (function of the natural wavelength profile) at height z

Corby & Wallington (1956) and Wallington (1958) describe these factors in detail, but here we need know only the units in which these factors are described:

A_Z is in km
M_H is in km
M_W is in km
U_G and U_R are in km/hr
N_Z is in km^{-1}

2.4 Vertical Velocity

The relationship between the vertical component of the airflow through a wave and the amplitude, lee wavelength and wind speed is:

$$V_z = 2\pi A_z U_z/W = (2\pi)\, M_H M_W U_G A_z/W \quad (4)$$

where V_Z = Maximum vertical speed in km/hr
and W = The lee wavelength in km.

2.5 Turbulence

Scorer's lee wave theory is based on a perturbation approach wich assumes that displacements in the flow are small enough for their squares and products to be neglected. This approximately tends to become invalid as the magnitude of the rate of change of amplitude with height increases. However, it appears likely that although calculated streamlines and derived airflow speeds may well be erroneous at least in regions of the flow where the small perturbation assumption is suspect, it is in these regions that the assumed smooth flow is likely to break down into turbulence – which has occasionally been found to be violent in strong lee wave flows. Holmboe & Klieforth (1957) describe such turbulence. Scorer has suggested that such turbulence is likely to be associated with the condition $|\partial A_z/\partial z| > 1$. The two factors that vary with height in the amplitude, A_z, are $1/U_z$ and N_z. Maximum values of $|\partial N_z/\partial z|$ in the three-layer model occur at the upper and lower boundaries of the middle, stable layer, and at points of inflexion, if any, in the amplitude profile within this stable layer.

2.6 Double Wave Systems

Some airstreams provide conditions favourable for two or more lee waves to co-exist. The shortest wave normally has a single amplitude maximum in the lower half of the troposphere. The next longer wave has two, usually unequal, maxima with a nodal surface between the levels at which these maxima occur. The lower level maximum is usually small, while the upper level maximum, usually in the upper half of the troposphere, is more significant, but both these maxima are usually less than that of the shorter, lower wave. Longer waves with more than two amplitude maxima rarely appear to be significantly large. The study described in this paper deals only with the lower tropospheric single maximum type of lee wave, but the principles and techniques enunciated could be applied to the longer double maxima type.

2.7 Upper Boundary Conditions

The model used in this paper assumes that the lee wave amplitude decreases to become negligible at high levels, which in this context is taken to be near and above the tropopause. Wallington & Portnall's (1958) computations suggest that in some occasional situations lee wavelengths and wave amplitudes are dependent on higher level boundary conditions which are not known and cannot be confidently assumed. This study does not attempt to make any advances in dealing with very high level boundary problems; it is assumed that amplitudes and lee wavelengths are not affected by conditions above the tropopause.

Study Procedure

3.1 Parameters

A three-layer atmosphere of the type already described can be specified by the following parameters:

The depth of the lowest, adiabatic, layer.. H_L
The depth of the middle, stable, layer.. H_S
The natural wavelength of the middle, stable, layer .... W_S
The natural wavelength of the uppermost, less stable layer. .... W_U

Values of the wind speeds (U_G and U_Z) at ground level and at a given height z must also be specified for vertical speed calculations.
With these parameters it is possible to calculate (as described in the paper by Wallington):
The lee wavelength.......................... W
The maximum natural wavelength amplitude factor (in the stable layer) ...... N
The vertical speed component at the level of this maximum amplitude factor V
A turbulence indicator taken as the maximum of $|\partial N_z/\partial z|_{\text{maximum}}$.
It is also possible to calculate the heights at which these maxima and the values of V occur, but these heights are of lesser importance in the general prediction problem. Assumption of the nature of the amplitude profile, its relationship to the stable layer, together with an indication of the magnitude of the items just listed will be enough for most practical purposes.
To reduce the number of parameters, all of the items listed above are scaled to H_S, the depth of the stable layer. Thus, the working data parameters become:

$$\frac{H_L}{H_S},\ \frac{W_S}{H_S},\ \frac{W_U}{H_S}$$

which can be used to compute

$$\frac{W}{H_S},\ NH_S,\ \frac{V}{H_S},\ \text{and } TH_S$$

where $T = |\partial N_z/\partial z|_{\text{maximum}}$.

Most long mountain ridges are asymmetrical in that their windward and leeward mean slopes are not equal. In this study a two-dimensional mountain ridge profile is taken to be that depicted in Fig. 2, the parameters being: the height of the crest, and ratio of upwind to downwind widths. To reduce the number of parameters needed to study relationships between the mountain factor and wavelength and mountain width, the wavelength and mountain width factor are scaled to the mountain width, i.e. values of W/W_M are used to compute M_W/W_M, W_M being the total mountain width.

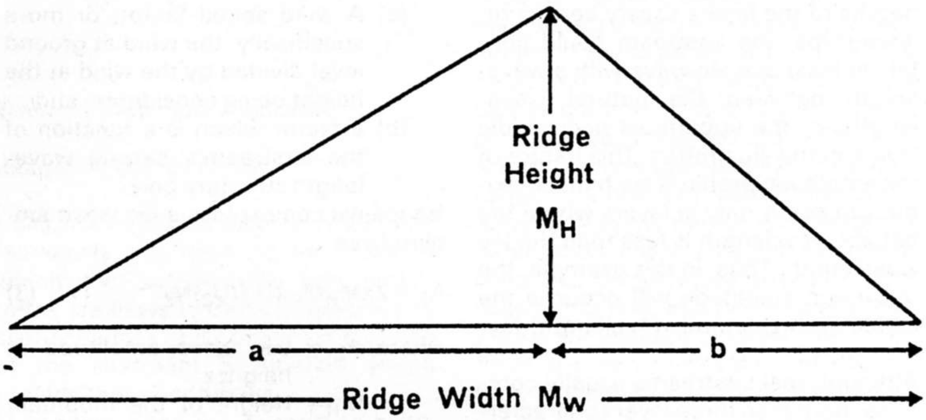


Figure 2
The simple 2-dimensional mountain ridge profile used in this study.

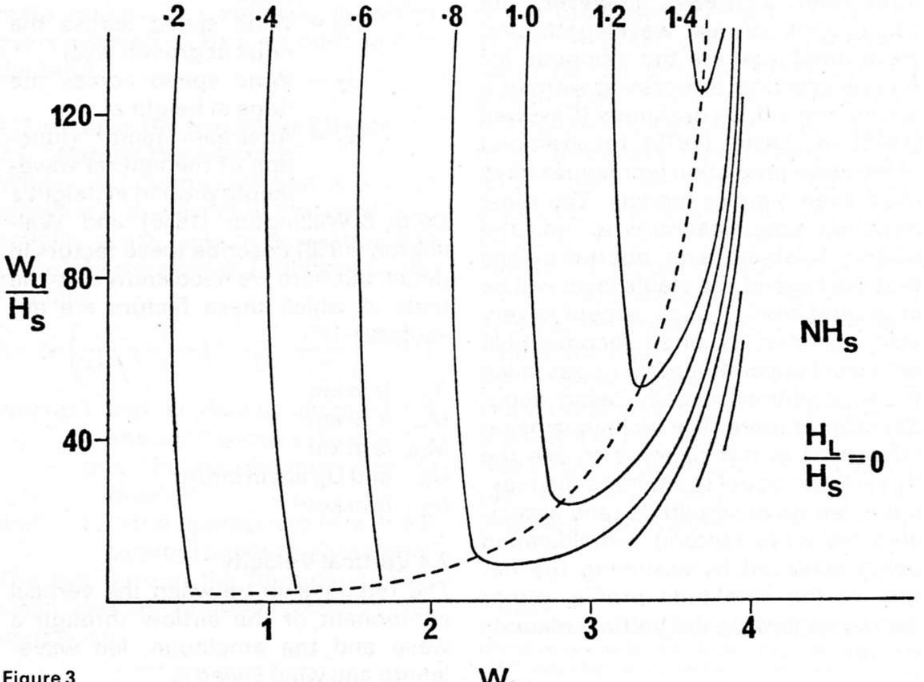


Figure 3
Curves illustrating the variation of NH_S with W_U/H_S and W_S/H_S for three values of H_L/H_S. To the left of the broken lines NH_S is almost independent of W_U/H_S; while to the right of the broken lines the amplitude factors are very sensitive to changes in airstream conditions.

The purpose of the scaling is, of course, to enable the derived airstream and mountain factors to be plotted against the minimum number of essential data parameters.

3.2 Computations
Values of:

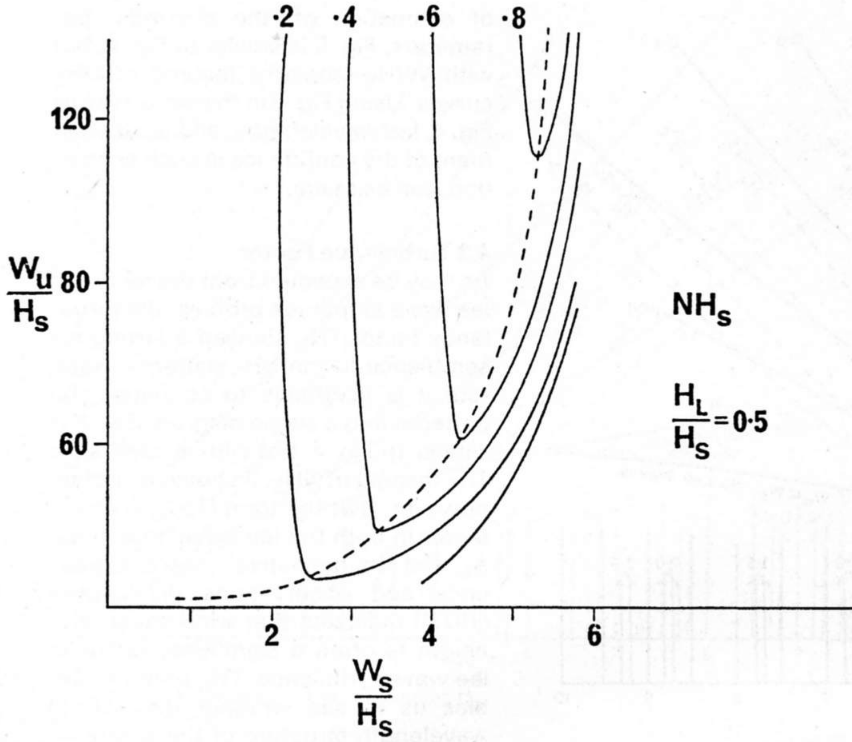
$$NH_S,\ \frac{W}{H_S}\ \text{and } TH_S$$

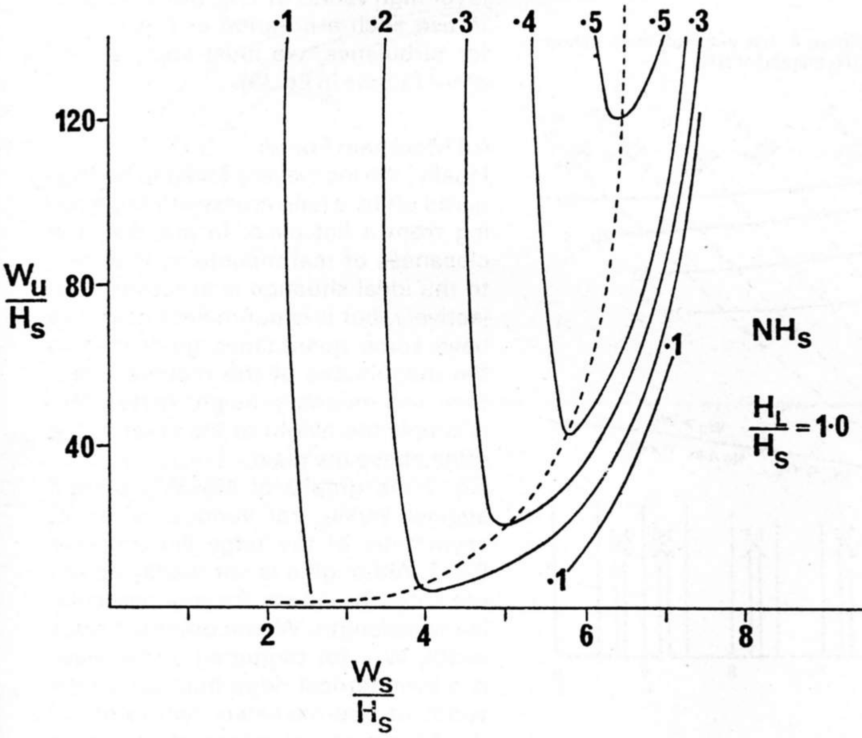
where computed and plotted for a selection of values of:

$$\frac{W_S}{H_S},\ \frac{W_U}{H_S}\ \text{and}\ \frac{H_L}{H_S}$$

Some vertical speed factors were computed, but they are not presented here as they can be derived from the other computed parameters and wind data.
Values of M_W/W_M were also computed and plotted against W/W_M for various ratios of the upwind to downwind widths of slopes of the mountain ridge.

4. Results of the Computations
4.1 Natural Wavelength Factor
Fig. 3 (a, b & c) are plots of NH_S against W_S/H_S and W_U/H_S for H_L/H_S = 0, 0.5 and 1.0. Computations and plots for other values of H_L/H_S were made, but





results for the three values illustrated here are enough to show the nature of the results. The general characteristics revealed are:

(I) For each value of H_L/H_S there is an upper limit of W_S/H_S, beyond which the theoretical conditions for lee waves are not satisfied. This upper limit for W_S/H_S increases, not linearly, from 4 for $H_L/H_S = 0$ to approximately 10 for $H_L/H_S = 3$, and to higher values for greater H_L/H_S. For low values of W_U/H_S, conditions for lee waves are not satisfied for smaller values of W_S/H_S than those just mentioned.

(II) For a wide range of values of W_S/H_S and W_U/H_S, NH_S is almost independent of W_U/H_S.

(III) For a narrower band of values of W_S/H_S, between the broken lines sketched in Fig. 3 and the upper limits of W_S/H_S for wave conditions, NH_S is very sensitive to slight changes in (or errors of estimation of) W_S/H_S, the range of N_SH_S ranging from maxima to zero with small increases in W_S/H_S.

(IV) The patterns for $H_L/H_S = 0, 0.5$ and 1.0 are of similar character and shape. They, and others computed for other values of H_L/H_S, all show the three characteristics already described, but the magnitudes of NH_S generally decrease, not linearly, with H_L/H_S.

Before discussing the implications of these patterns, it may be useful to note that dividing NH_S by H_S to obtain N shows the sensitivity of N to the airstream parameters (over a band of values near the limits for waves) to be even greater than that illustrated in the diagram. Although this sensitivity means that we cannot devise a quick, simple prediction technique for the sensitive band, we can at least be aware of the band devise guidelines to indicate whether or not a prediction of N can be made with confidence.

Fig. 4 shows two sets of isopleths plotted against values of W_S/H_S and W_U/H_S. The broken lines correspond to the broken lines sketched in Fig. 3 and similar lines for other values of H_L/H_S not illustrated in Fig. 3. The full-line curves denote values of NH_S for values of W_S/H_S and W_U/H_S on the H_L/H_S curves. Approximate values of NH_S for a three-layer model can be obtained as follows:

(I) From given, predicted or assumed temperature and wind profiles that appear to approximate to the three level structure, calculate W_U, W_S, H_S and H_L.

(II) Note the position of the point (W_S/H_S, W_U/H_S) on the diagram shown in Fig. 4.

Then:

(a) If this point is on or to the left of the H_L/H_S curve the required value of NH_S will be within about 5% of the NH_S value corresponding to the given values of W_S/H_S and H_L/H_S.

(b) If the given value of W_S/H_S is greater than the wave limits for W_S/H_S marked for values of H_L/H_S in the bottom section of the diagram, or if the point (W_S/H_S, W_U/H_S) is below the curve

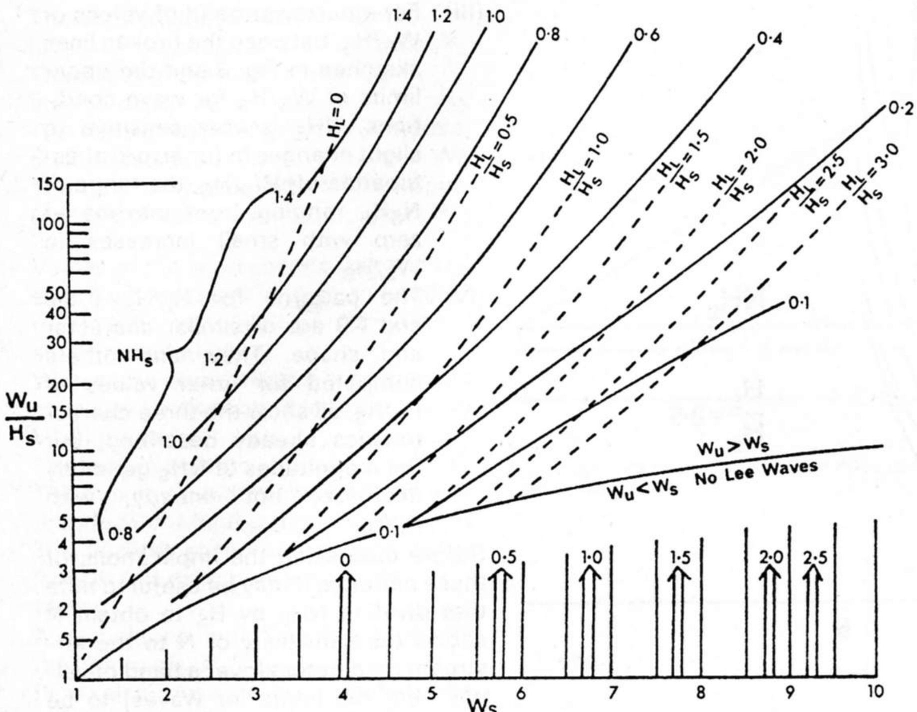
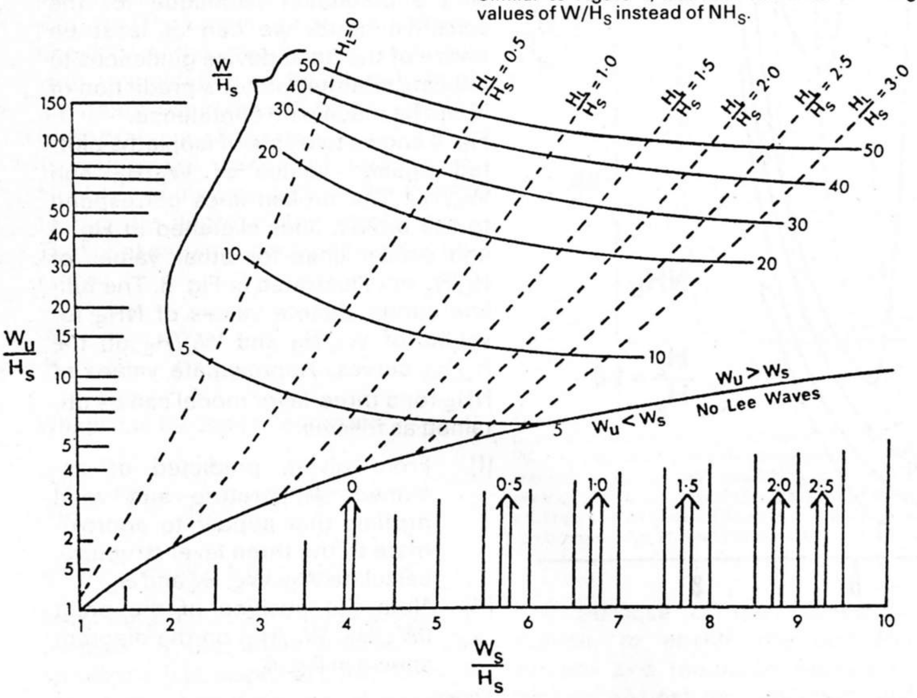


Figure 4
The full-line curves show values of NH_S for values of W_S/H_S and W_U/H_s on the H_L/H_s curves. The derivation of approximate values N/H_S for a 3-layer model is described in the text.

Figure 5
Similar to Figure 4, but with full lines denoting values of W/H_S instead of NH_S.



$W_S = W_U$, then stationary lee waves will not exist.

(c) If the point (W_S/H_S, W_U/H_S) is between the given H_L/H_S curve and the limits for lee wave conditions described in (b), above, lee waves will exist but determination of NH_S will be difficult to make with confidence. These values will decrease quickly from the NH_S values in the H_L/H_S curves, as the point (W_S/H_S, W_U/H_S) moves from the H_L/H_S curve towards the wave condition limits.

4.2 Lee Wavelength

Lee wavelength patterns are similar to those for the natural wavelength factor, in that, for a wide range of values of W_S/H_S and W_U/H_S, values of W/H_S are almost independent of W_U/H_S, while, in a narrow band towards the limit of lee wave conditions, W/H_S is sensitive to small changes in, or errors of estimation of, the airstream parameters. Fig. 5 is similar to Fig. 4, but with W/H_S isopleths instead of NH_S curves. Using Fig. 5 in the same way as Fig. 4, lee wavelengths, and an assessment of the confidence in such estimation, can be made.

4.3 Turbulence Factor

As may be expected from the nature of lee wave amplitude profiles, the turbulence factor TH_S showed a strong resemblance to the NH_S patterns. Here, too, it is justifiable to condense the patterns into a single diagram. Fig. 6 is similar to Fig. 4, but with isopleths for TH_S instead of NH_S. It should be noted, however, that the term $(1/U_z)$ is also a factor in both the lee wave amplitude, A_z, and the derivative, $\partial A_z/\partial z$. Experience and observations (Wallington (1977)) indicates that wind shear with height is often a significant factor in lee wave turbulence. TH_S pattern enables us to see whether the natural wavelength structure of the airstream favor high values of TH_S, but if we wish to use such a criterion as $|\partial A_z/\partial z| > 1$ for turbulence, we must apply all the other factors in Eq. (3).

4.4 Mountain Factor

Ideally, the lee wave is taken to be triggered off by a long crosswind ridge rising from a flat plain. In practice, the closeness of real mountainous terrain to the ideal situation is assessed subjectively, but it is nontheless useful to have some quantitative guidelines to the magnitudes of the mountain factors. The mountain height factor, M_H, is simply the height of the crest of the ridge above the plain.

Fig. 7 are graphs of M_W/W_M plotted against W/W_M for various values of asymmetry of the ridge illustrated in Fig. 2. Although it is not readily apparent from the curve, for any particular lee wavelength, W, the optimum ridge width, W_M, for triggering of the wave is a symmetrical ridge that has a total width of approximately two-thirds of W. Asymmetrical ridges of the same total width are less efficient in the range $W/W_M < 3$, but marginally more efficient at higher values of W/W_M.

5. Conclusions

This numerical study shows that the behaviour of lee wave factors enables approximate prediction methods to be derived for lee wave characteristics in a three-layer lee wave model. Such predictions cannot be made with confidence in all airstream conditions, but

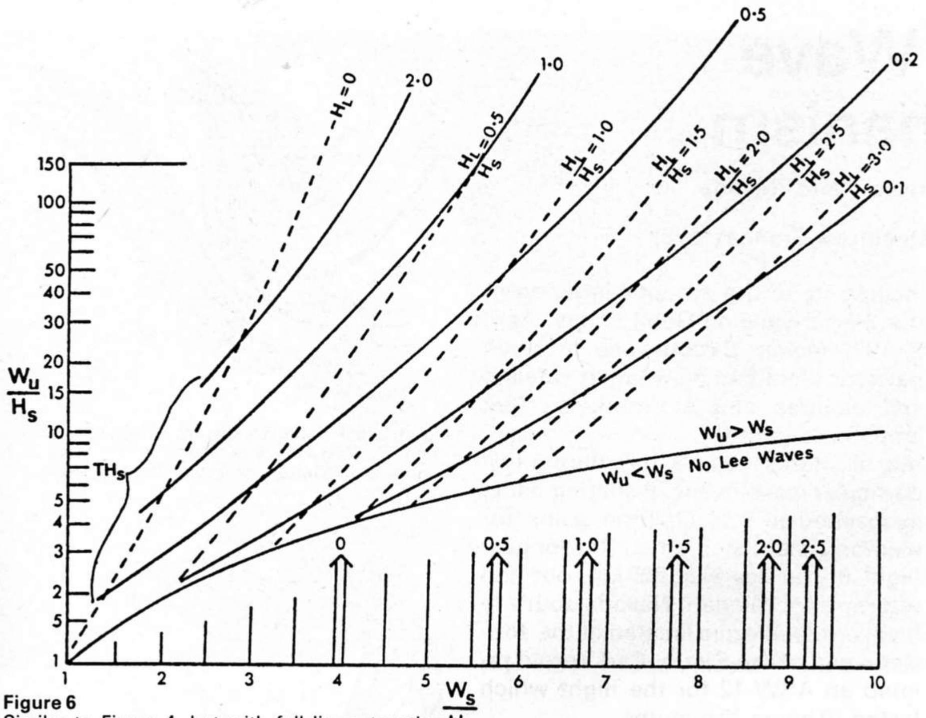


Figure 6
Similar to Figure 4, but with full lines denoting values of TH_S instead of NH_S.

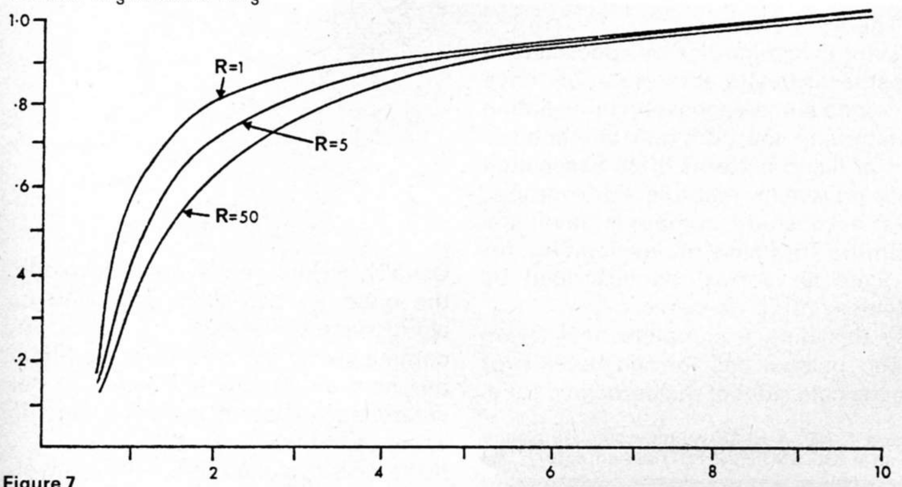


Figure 7
Graphs M_W/M_M plotted against W/W_M for various values of asymmetry of the ridge illustrated in Figure 2.

at least the zones of easy and difficult predictive conditions can be recognised and assessments of confidence in predictions of maximum lee wave amplitude, lee wavelength maximum vertical velocity and risk of turblence can be given.

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