Optimal Flight Strategy in a given Space-Distribution of Lifts with Minimum and Maximum Altitude Contraints

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The following is an abridged version of Université de Liège paper SART 38/3 with the mathematical analysis omitted. For the full treatment the original paper should be consulted. – Editor.

1. Introduction

In recent years several researchers examined the many various problems connected with the optimization of the flight strategy for a sailplane soaring cross-country. Some of the most significant works can be found in [1]–[10]. From the point of view of the theorist in optimization, those problems have a perfectly well defined objective. Indeed, it is always the total time which must be minimized, either because the sailplane is involved in a contest or because, when lift arises only from thermals, the useful time for distance flight is limited to the day period during which the sun produces thermals. On the other hand, an aspect of the problem which does not seem to be completely solved at the present time is the atmospheric model that should be used. Some attempts [11]–[12] have been made to model the shape of a thermal. We shall not be concerned with that difficult problem because we shall assume concentrated lift. This assumption is justified in the framework of the problem that we are treating.

The problem that we are examining in this paper is completely deterministic: no stochastic aspects are considered. We are essentially concerned with the problem of determining, for a given sailplane, the optimal flight strategy which corresponds to travelling a given distance in minimum time with zero net altitude loss. We assume that lifts are concentrated at some given places unequally spaced along the trajectory. The locations of those lifts as well as their characteristics are constant with time; strengths of the various lifts are not generally equal. We suppose that the air mass in between lifts is still, i.e. there is no sinking zone surrounding the core of the thermal. Finally the flight must stay within two given flight levels: the lowest one corresponds to safety with respect to ground and the highest one means that no cloud flying is allowed. Thus, the sailplane's flight is divided into steps, each of which consist of an ascent in a thermal and a glide at constant speed to the next thermal. The flight starts and ends at the given minimum flight level. We assume that there is no wind. The projection of one glide on the ground is rectilinear, however the projections of the glides of successive steps can be inclined to one another. The pilot must take sequentially two kinds of decisions: how much to climb in each lift and what speed to adopt in between two lifts.

By its very definition, the problem prohibits soaring with so called «dolphin» or «essing» mode. We are pretty well aware, as was rightly pointed out in [2] that ... «It is generally recognized that many of the very fast cross-country flights achieved in recent years have been made under conditions where the latter two modes were utilized and relatively little time was spent in thermaling» ... However we still think that the problem treated here is of interest: first because the atmospheric conditions do not always allow for the «dolphin» or «essing» modes and second because, to the best of our knowledge, it is the first time that a problem is solved which involves not just one step but a whole flight taking also into account altitude constraints.

One last remark is that although in general the results cannot be used by a pilot on an actual flight since he should know the characteristics of all the thermals that he will encounter later, one still can used those results for many purposes. We will just mention here two applications: first, simulation experiments can be conducted with competition pilots to enable them to compare their strategies with the optimal one and second, performances of sailplanes can be compared with respect to a given standardized space-distribution of lifts.

2. Statement of the Problem

As was said in the introduction, we divide the sailplane's flight into steps, each of which consists of an ascent in a thermal at minimum sinking rate followed by a glide at constant speed to the next thermal.

The situation is best illustrated in figure 1 which is a drawing in a vertical plane: the positive direction of the y axis indicates the travelling direction of the sailplane and the vertical z axis is positive upwards. Accordingly, all horizontal speeds will be considered positive to the right and all vertical speeds will be considered positive upwards.

For reference, the minimum altitude is taken equal to zero and the maximum altitude is denoted by h.

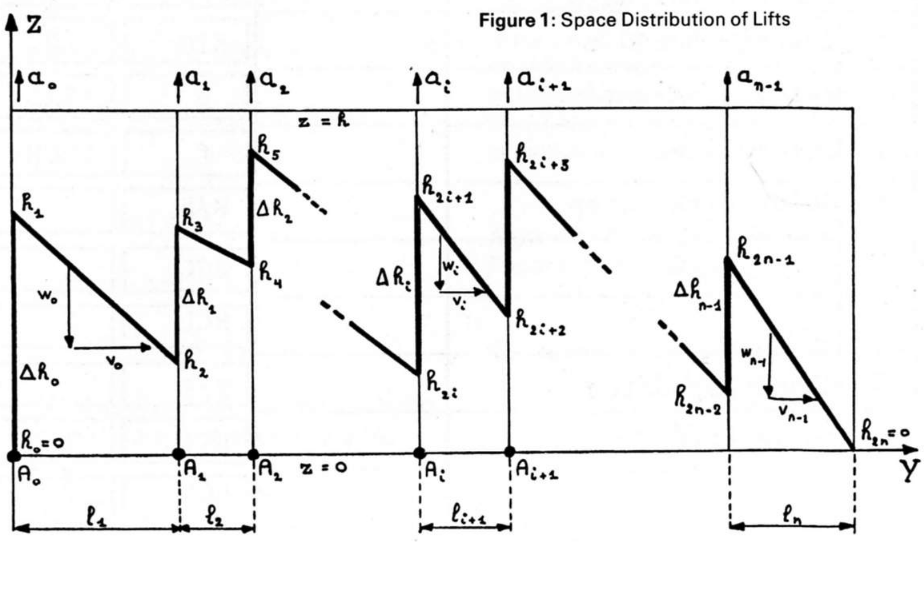


Figure 1: Space Distribution of Lifts

The sailplane starts at point A_0 corresponding to $y = 0$ and to altitude $h_0 = 0$, then climbs into the first thermal at an achieved rate of climb a_0. It leaves the first thermal at altitude h_1 and then glides at constant forward speed v_0 (corresponding to sinking rate w_0) to the second thermal which is reached at altitude h_2, having travelled the horizontal distance l_1. Generally, the sailplane reaches the i-th lift with a forward speed v_{i-1} and at an altitude h_{2i+1} with an achieved rate of climb a_i and leaves the thermal with a forward speed v_i. Finally the sailplane must reach the point A_n of coordinates $y = 1$ and $z = 0$. Of course, for the problem to make sense, all distances l_i must be such that they can be achieved by the sailplane flying at the speed of maximum Lift-to-Drag ratio with a loss of altitude at most equal to h.
Remember that we assumed that there is no wind. Also we shall neglect the transient dynamical effects occurring when entering or leaving a thermal, hence the only characteristic of the sailplane that we shall need will be the polar equation relating the forward speed v to the sinking rate w.
One last thing we must discuss before writing down the equations is the vertical characterizations $a_i(z)$ of the thermals. We shall examine two cases:

a) the strength of the thermal is constant with the altitude ($a_i = a_i(z) =$ constant),
b) the strength of the thermal at first increases with altitude up to a maximum and then decreases ($a_i = a_i(z)$).

The pilot has two controls at his disposal:
a) $\triangle h_i$: the gain of altitude in the i-th thermal
b) v_i: the speed to fly after leaving the i-th thermal.

If we further denote by t_{2i} the time to climb in the i-th thermal and by t_{2i+1} the time to travel the distance l_{i+1}, we can write the following relations:
Finite difference equations for the altitudes:

$$\left.\begin{aligned} h_{2i+1} - h_{2i} &= \triangle h_i \\ h_{2i+2} - h_{2i+1} &= \frac{w_i(v_i)}{v_i} \cdot l_{i+1} \end{aligned}\right\} \quad (1)$$

where $i = 0, 1, 2, \ldots, n-1$ and $w_i(v_i)$ is given by the polar equation.

Cost:

$$\left.\begin{aligned} t_{2i} &= \int_{h_{2i}}^{h_{2i}+\triangle h} \frac{dz}{a_i(z)} \\ t_{2i+1} &= \frac{l_{i+1}}{v_i} \end{aligned}\right\} \quad (2)$$

Control constraints:
$\triangle h_i \geqq 0, \qquad i = 0, 1, 2, \ldots, n-1 \quad (3)$
Initial and terminal constraints:
$h_o = 0; h_{2n} = 0 \quad (4)$
Altitude constraints:
$h_{2i}+1 \leqq h; h_{2i} \geqq 0 \quad (5)$

The problem is thus to find the optimal strategy, i.e. the sequence(s) $\triangle h_0$, v_0, $\triangle h_1$, v_1, ..., $\triangle_{n-1}$, v_{n-1}, which among all such sequences satisfying the relations (1) and (3), (4) and (5), minimize(s) the total cost (i.e. the total time):

$$T = \sum_{i=0}^{n-1} \left[\int_{h_{2i}}^{h_{2i}+\triangle h_i} \frac{dz}{a_i(z)} + \frac{l_{i+1}}{v_i} \right] \quad (6)$$

obtained by summing all the partial costs (i.e. partial times given by (2).
There may be more than one minimizing sequence but the minimum cost either is unique or does not exist.
As such the problem is of course one of mathematical programming with equality and inequality constraints. However, the way we have set it up, it is in fact a «discrete optimal control problem».

3. Solutions

Application of the method of [13] leads to the following solutions.

3.1. Thermals of Constant Strength

First of all, in the case of thermals of constant strength, it is easy to show that the hypothesis of directional convexity is satisfied provided only that the polar w(v) is a concave function. Indeed, the time to climb in thermal a_i becomes:

$$t_{2i} = \frac{\triangle h_i}{a_i} \quad (7)$$

No altitude constraints.
We can distribute at will the total necessary climb $\triangle h$ into the different thermals of maximum strength; we do not climb at all in the other thermals.
The total time (6) is:

$$T = \frac{\Delta h}{a_M} + \frac{l}{v_0} \quad (8)$$

where $a_M = \max_i (a_0, a_1, \ldots, a_{n-1})$.

with of course:

$$\frac{\triangle h}{l} = -\frac{w(v_0)}{v_0} \quad (9)$$

so that the cross country speed v_{cc} is:

$$v_{cc} = \frac{l}{T} = \frac{a_M}{\frac{d}{dv} w(v_0)} \quad (10)$$

This quite obvious and trivial result is of course of the Mac-Cready type so that a_M and v_{cc} can be interpreted graphically as shown in figure 2.

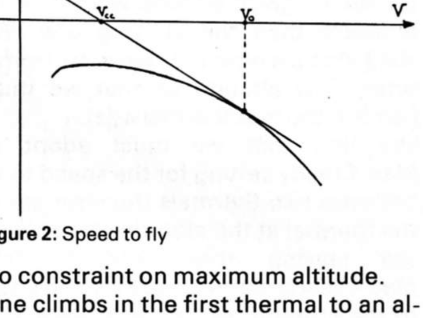


Figure 2: Speed to fly

No constraint on maximum altitude.
One climbs in the first thermal to an altitude which allows reaching at the minimum altitude – flying at a «Mac-Cready speed» relative to that first lift – the first of the next thermals which is stronger than the first one. One then climbs in that thermal and repeats the process to the next stronger thermal ... and so on.
Where several thermals are of the same strength we do not change the total time by distributing at will the total necessary climb $\triangle h$ to reach a stronger thermal into the thermals of same strength.
The situation is illustrated in figure 3.

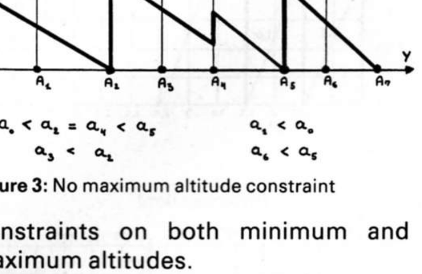


Figure 3: No maximum altitude constraint

Constraints on both minimum and maximum altitudes.
The speed to fly after a climb is of the Mac-Cready type relative to the last thermal unless the maximum altitude has been reached in that thermal. Two successive speeds can only differ when at least one of the boundary altitudes has been reached in between. In between two climbs, the speed to fly does not change whatever the strengths of the thermals encountered in between.

3.2. Thermals of Variable Strength with Altitude

We now examine the case where the thermal at first increases with altitude up to a maximum and then decreases: $a_i = a_i(z)$.

We shall not consider altitude constraints for this case because we shall assume that below a minimum altitude and above a maximum altitude the strength of the thermal is equal to zero so that the altitude constraints are implicit.

We shall consider as boundary conditions that we start from a given altitude to reach a given altitude which can be different than the starting one. We shall also assume that we start from a rather low altitude so that we must climb in the first thermal $a_0(z)$.

We find that we must adopt as Mac-Cready setting for the speed to fly between two thermals the strength of the thermal at the altitude at which we are leaving that thermal. That Mac-Cready setting must also be equal to the strength of the next thermal in which we climb at the altitude at which we are entering that thermal. We do not climb in thermals which would lead to the situation where we would have to go down in one thermal. When crossing a neglected thermal, the speed to fly does not change. In the last thermal in which we climb, we actually climb to an altitude such that by adopting a Mac-Cready setting equal to the strength of the thermal at that altitude, we reach the desired final altitude.

4. Examples

As a simple example, we have taken the 300 km flight schematized in figure 4. The lifts are equidistant (10 km) for simplicity although it is by no means implied in the preceeding rules for optimality. The lift strengths are indicated in m/sec along the y axis. They increase progressively during the flight, then decrease but are in general unequal. The altitude limits are 0 and 1000 m. We consider a sailplane having a polar equation given by

$w = -0.0016409\ v^2 + 0.061637\ v - 1.02557$

The optimal strategy for that lift distribution is illustrated in figure 4 where the Mac-Cready setting for each glide is indicated. It follows as a simple and systematic application of rules for optimality established above. Note that the flight strategy consists in hitting systematically the altitude constraints, but at the km 110 and at km 170 where we gain, in a lift equal to the present Mac-Cready setting, the altitude just necessary for reaching at zero altitude the next best lift. Note also that the Mac-Cready setting is not always equal to the strength of the next lift used. Note finally that this example clearly justify the practical rule of flying «low» where the lifts are improving and flying «high» when they are deteriorating. The total time required for the flight is T = 13331 sec corresponding to a cruising speed of 81.01 km/h.

To illustrate and quantify on the same example the importance of the global flight strategy, we have compared the result obtained by a pilot flying according to the following rules.

Decision to use a lift:

- from h = 0 to h = 300 m take any lift $a_1 > 0$
- from h = 300 m to h = 600 m take a lift only if $a_i \geqq 1$ m
- from h = 600 m to h = 1000 m take a lift only if $a_i \geqq 2$ m

Altitude gained

- climb up to 600 m if the lift is $a_i < 2$
- climb up to 1000 m if the lift is $a_i \geqq 2$ m.

Speed to fly

Adopt a Mac-Cready setting corresponding to the moving average of the last 3 lifts encountered (even if they are not used).

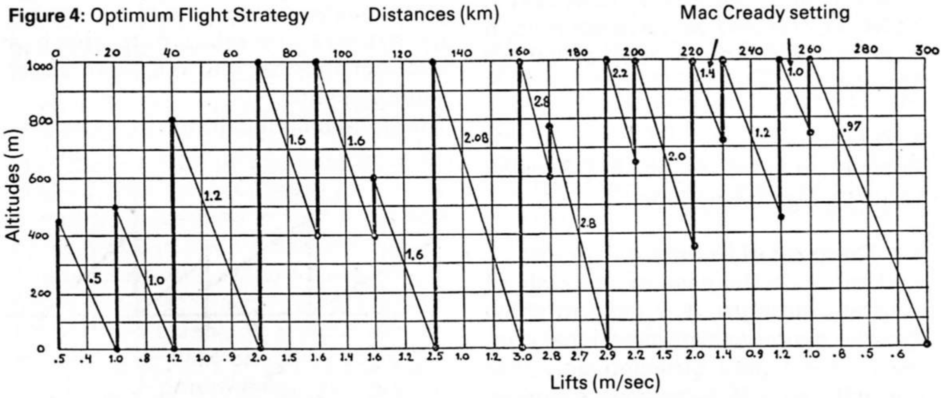
The result is illustrated in figure 5 and leads to a total time of (= 14401 sec (cruising speed of 74.9 km/h).

5. Conclusions

Simple rules have been given for finding the global optimal flight strategy in the case of unequally spaced lifts of variable strength taking into account altitude constraints. The assumption that the locations and strengths of the lifts are known in advance evidently makes the results of questionable in-flight practical usefulness. However it is now possible to determine optimal flight strategies in a set of given situations that are often encountered during a flight. The importance of giving due consideration to the altitude constraints is evident. From various tests achieved by the authors in a community of experienced competition pilots it appears that the rules given here are, at best, intuitively approximated.

Improvements to the theory should take into account the size and structure of the thermals in order to allow for dophin flight segments. This seems possible only if a numerical model is set up. It implies that no simple rules for optimality will be obtained in that case but that a catalogue of optimal strategies in a given set of situations could be derived.

Figure 4: Optimum Flight Strategy



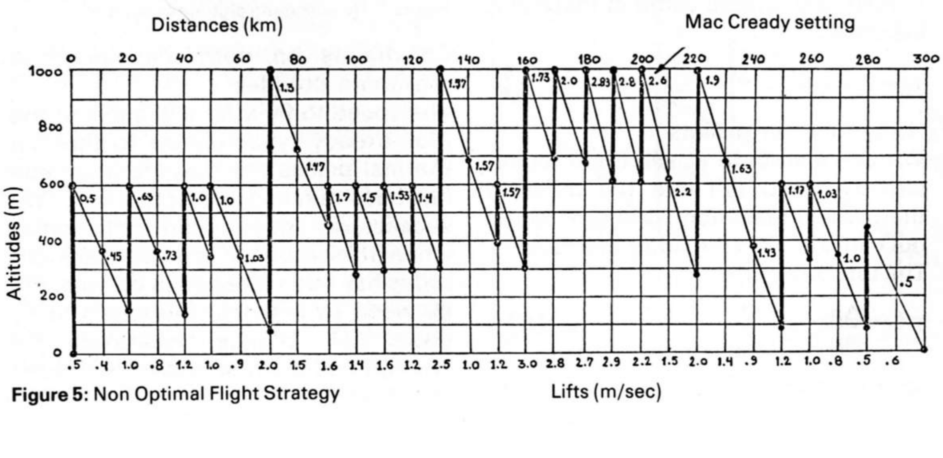


Figure 5: Non Optimal Flight Strategy

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