Distance Estimation Error and Stationary Optimal Gliding

Dr. Risto Arho
Tampere University of Technology, Tampere, Finland

In this article we divide a sailplane's flight into steps, each of which consists of an ascent in a thermal and a glide to the next thermal (see Fig. 1). We assume that there is no wind. The projection of one glide on the ground is rectilinear, however the projections of the glides of successive steps can be inclined to one another.

Our pilot wants to fly a distance in minimum time and maintain a constant base level of flight, that is, every ascent is started at the same altitude (see Fig. 1). Generally speaking the allowance of varying base level might be necessary in finding the optimal way of flight which absolutely minimizes the flight time. The solution of the problem with varying base level could yield an optimal solution, where the flight policy of each step would depend on the thermal conditions of all the steps ahead. However the pilot cannot be fully aware of the thermal conditions far ahead so that this kind of treatment of the problem is not adequate. On the other hand, if the base level is kept constant and thermal conditions along the course are stationary (i.e. thermals maintain their strengths constant and do not move, which approximations are of course generally quite coarse) and the thermals reach high enough, then in order to fly the course in minimum time the pilot has simply to minimize the flight time of each separate step, as we shall see. Thus the pilot after having entered the thermal of the i:th step on base level observes the c_i and estimates the distance to the next thermal he will use (s_i). Perhaps there are several candidates for the next thermal. It is impossible for the pilot to pick up the absolutely optimal one. At this point he has to resort to intuition. This is why we are not seeking for the absolute minimum but 'a kind of minimum'. The parameters c_i and s_i as well as the gliding characteristics of the sailplane uniquely determine the optimal flight policy of the step. All possible thermals that are met during the glide are ignored. Although this step by step optimization procedure generally does not yield the absolute minimum, it often however optimizes that which in practice can be optimized.

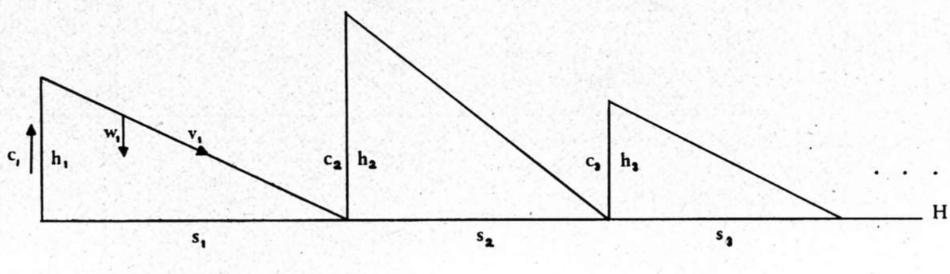

Figure 1. Steps of flight. c = ascent velocity in thermal, s = distance between two successive thermals that are used, h = altitude increase in thermal, v = gliding velocity, w = sinking rate, H = base level of flight.

1. Optimal flight

We first consider the optimal flight of one step. Let us assume that the sailplane's polar equation is of the form – or can be approximated by

$$w = Av^3 + Bv, \tag{1}$$

where A and B are constants. This is a good approximation in the region of the 'laminar-bucket' and is thus valid during gliding. If t_1 denotes the ascent-time and t_2 = the gliding time, we have

$$h = ct_1,\ s \approx vt_2,\ h/s \approx w/v. \tag{2}$$

From Eq. (1) and Eqs. (2) we write

$$T = t_1 + t_2 = \frac{s}{c}\,(Av^2 + B) + \frac{s}{v} = T\,(v). \tag{3}$$

The relative optimum of T (v) is obtained at

$$v_{opt} = \sqrt[3]{\frac{c}{2A}}. \tag{4}$$

$$\text{As } \frac{d^2T}{dv^2}\,(v_{opt}) = \frac{6As}{c} > 0\ (A > 0),$$

v_{opt} stands for the velocity that minimizes the total flight time of the step. The optimal altitude increase in the thermal is

$$h_{opt} = s\left(\frac{w}{v}\right)_{opt}, \tag{5}$$

where by Eqs. (1) and (4)

$$\left(\frac{w}{v}\right)_{opt} = A\left(\frac{c}{2A}\right)^{2/3} + B. \tag{6}$$

Eq. (4) gives the McCready-velocity of a sailplane, whose polar is given by Eq. (1). We can derive this analytical result also (see Fig. 2) by dropping a tangent from the point v = o, w = –c to the polar. The derivation of Eq. (4) in this way proceeds as follows: From Eq. (1) we have

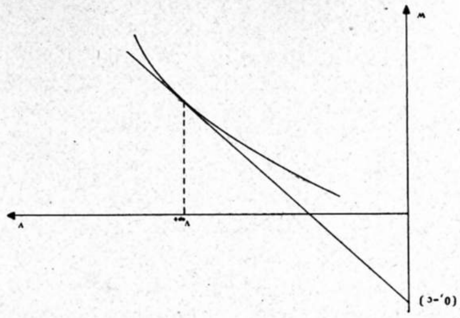

Figure 2. Graph of polar. v_{opt} = optimal gliding velocity, which we call the McCready-velocity.

$$\frac{dw}{dv} = 3Av^2 + B \tag{7}$$

and consequently the gradient of the tangent is $3Av^2_{opt} + B$. Thus the equation of the tangent that goes through the point (o, –c) is

$$w + c = (3Av^2_{opt} + B)\ v. \tag{8}$$

But the tangent goes also through the point (v_{opt}, w_{opt}) and so we obtain

$$Av^3_{opt} + Bv_{opt} + c = (3Av^2_{opt} + B)\ v_{opt}, \tag{9}$$

from which the result of Eq. (4) at once follows.

We next turn to the optimal flight of several (n) steps. Now the total flight-time of the i:th step

$$T_i = \frac{s_i}{c_i}\,(Av_i^2 + B) + \frac{s_i}{v_i} \tag{10}$$

is a function of v_i only, if s_i and c_i are assumed to be constants, that is $T_i = T_i\,(v_i)$. Then the time needed to fly through the course of n steps is

$$\sum_{i=1}^{n} T_i\,(v_i) = T_1\,(v_1) + \ldots + T_n\,(v_n) = f\,(v_1, \ldots, v_n). \tag{11}$$

In order to minimize f ($v_1, \ldots, v_n$) we write

$$\frac{\partial f}{\partial v_1} = \ldots = \frac{\partial f}{\partial v_n} = 0 \tag{12}$$

or

$$\frac{dT_1}{dv_1} = \ldots = \frac{dT_n}{dv_n} = 0. \tag{13}$$

In other words we have to optimize every step separately.

2. Distance estimation error

Up to this point we have assumed that the pilot does not make errors in estimating s. We now suppose that he makes an error ds, which is assumed to be normally distributed with mean m (s) and standard deviation σ (s) (mean and s.d. of ds of course depend on the distance to be estimated).

As $\left(\frac{w}{v}\right)_{opt}$ does not depend on s [see Eq. (6)], the correct glide and the glide performed by the pilot are equally steep, that is $h_o/s_o = h/s$ (see Fig. 3). Then obviously the altitude error

$$dh = \left(\frac{w}{v}\right)_{opt} ds. \tag{14}$$

It consequently follows that dh is normally distributed with mean $\left(\frac{w}{v}\right)_{opt}$ m (s) and standard deviation $\left(\frac{w}{v}\right)_{opt}$ σ (s).

We next suppose that having entered the next thermal too high, as in Fig. 3 or too low the pilot starts a new step on a new base level of flight that differs from the preceding one by dh. In Fig. 3 he thus starts the new ascent at point P and if his estimate of the new distance were faultless, he would finish the new optimal glide on the level of point P. If his estimate however is erroneous, he will enter the next thermal above or below the level of point P. Acting in this way (no base level corrections) his base level of flight after n steps differs from his initial level by

$$\Delta h = \sum_{i=1}^{n} dh_i. \tag{15}$$

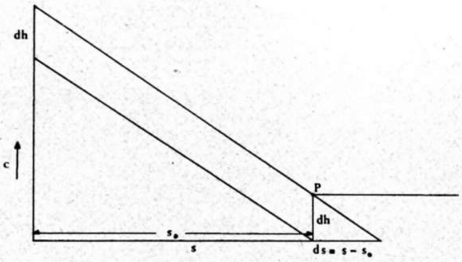

Figure 3. Erroneous step. s=distance estimated by pilot, s_o = actual distance, dh = altitude error due to distance error.

As the errors dh_i are normally distributed independent random variables, Δh is normally distributed and its mean and standard deviation are given by

$$m_{\Delta h} = \sum_{i=1}^{n} \left(\frac{w}{v}\right)_{opt_i} m(s_i) = \sum_{i=1}^{n} \left[A\left(\frac{c_i}{2A}\right)^{2/3} + B\right] m(s_i), \tag{16}$$

$$\sigma_{\Delta h} = \sqrt{\sum_{i=1}^{n} \left(\frac{w}{v}\right)^2_{opt_i} \sigma^2(s_i)} = \sqrt{\sum_{i=1}^{n} \left[A\left(\frac{c_i}{2A}\right)^{2/3} + B\right]^2 \sigma^2(s_i)}, \tag{17}$$

where use has been made of Eq. (6). We now consider a special case and assume (1) that the distances s_i are of the same magnitude = $\bar{s}$ and (2) that m ($\bar{s}$) = 0, which means that the pilot does not make a systematic error in estimating distances of magnitude $\bar{s}$. From Eqs. (16) and (17) then yield

$$m_{\Delta h} = 0, \tag{18}$$

$$\sigma_{\Delta h} = \sigma(\bar{s}) \sqrt{\sum_{i=1}^{n} \left[A\left(\frac{c_i}{2A}\right)^{2/3} + B\right]^2}. \tag{19}$$

Finally we give an example of the special case mentioned above: Let $\bar{s}$ = 10 km, m ($\bar{s}$) = 0, σ ($\bar{s}$) = 2 km, n = 10 and $c_1 = \ldots = c_{10}$ = 2 m/sec and let the sailplane be Nimbus-2. In the reference Uotila has approximated the polar of Nimbus-2 by Eq. (1). By making the relevant dimension changes in his coefficients we have

$$A = 1.106 \times 10^{-5}\ \mathrm{sec^2/m^2}, \quad B = 0.012. \tag{20}$$

From Eq. (19) we now compute $\sigma_{\Delta h} \approx 220$ m and by Eq. (18) $m_{\Delta n} = 0$. Thus Δh of this example is (0,220 m)-normally distributed. In other words: at the end of his course of 10 steps the pilot finds himself more than 220 m off (above or below) the initial base level of flight with the probability of 32%. This error is entirely due to his distance estimation errors. As σ (10 km) =2 km characterizes a pilot with quite a good distance estimation ability, we have by no means arrived at an exceptionally erroneous situation in the example.

In the light of our example we see that level corrections might be necessary every now and then along the course (compensation of Δh for instance in connection with the ascent of every fifth step) so that the base level of flight would not get too high or low in comparison with the altitude range of thermals.

Reference: Uotila, J.; Tasoituskertoimet purjelentokilpailuissa (Glider handicapping coefficients), diploma thesis, Helsinki University of Technology, 1972.