Elastic Properties of Plates Glued of Orthotropic Layers

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Presented at the 12th OSTIV Congress, Alpine, USA (1970)

Assumptions and Basic Relations

As a basic design element there has been assumed the layer consisting of straight parallel fibres (with no cross-binding fibres) with intermediate space filled with material of considerably lower elastic and strength coefficients than those for fibre. The filler is joined (glued) to the whole surface of the fibre.
Neglecting the difference of stresses between the filler and the fibres, such a material in the macroscopic sense can be treated as a homogeneous orthotropic body (the fibres of 0.01 mm thickness uniformly distributed) with main elastic axes directed along the fibres.
Restricting our consideration to a plane stress-system in the plane of the plate (fig. 1) we can describe the properties by means of four elastic coefficients:

E_x – young modulus along the fibres

E_y – young modulus across the fibres

G – shearing elasticity modulus

μ – poisson's number.

In the main elastic axes system there are the following relations:

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \frac{\mu\sigma_y}{E_x} \qquad (1)$$

$$\varepsilon_y = -\frac{\mu\sigma_x}{E_x} + \frac{\sigma_y}{E_y} \qquad (2)$$

$$\gamma = \frac{\tau}{G} \qquad (3)$$

The further considerations deal with a range of loadings where the elastic coefficients are constant and independent of the loading direction (tension or compression, shearing to right or left).
When considering the stresses and strains with respect to the axes inclined at angle φ to the fibre direction the formulae become some more complex (fig. 2) because the longitudinal stresses involve at the same time the shearing strains. Shearing in turn, involves the stretches or contractions.

Noting:

η_x – shearing influence coefficient on stretch along x' axis

η_y – shearing influence coefficient on stretch along y' axis

we obtain:

$$\varepsilon_x' = \frac{\sigma_x'}{E_x'} - \frac{\mu'\sigma_y'}{E_x'} + \frac{\eta_x'\tau'}{E_x'} \qquad (4)$$

$$\varepsilon_y' = -\frac{\mu'\sigma_x'}{E_x'} + \frac{\sigma_y'}{E_y'} + \frac{\eta_y'\tau'}{E_y'} \qquad (5)$$

$$\gamma' = \frac{\eta_x'\sigma_x'}{E_x'} + \frac{\eta_y'\sigma_y'}{E_y'} + \frac{\tau'}{G'} \qquad (6)$$

For orthotropic bodies the relations between the elastic coefficients with respect to the main elastic axes system and those for the system of axes inclined at angle φ are as follows:

$$E_x' = \frac{4\,E_x E_y G}{4G\,(E_x \sin^4\varphi + E_y \cos^4\varphi) + E_y\,(E_x - 2\mu\,G)\sin^2 2\varphi} \qquad (7)$$

$$E_y' = \frac{4\,E_x E_y G}{4\,G\,(E_x \cos^4\varphi + E_y \sin^4\varphi) + E_y\,(E_x - 2\mu\,G)\sin^2 2\varphi} \qquad (8)$$

$$\mu' = -\frac{(E_xG + E_yG - E_xE_y)\sin^2 2\varphi - 4\mu\,GE_y\,(\sin^4\varphi + \cos^4\varphi)}{4\,G\,(E_x \sin^4\varphi + E_y \cos^4\varphi) + E_y\,(E_x - 2\mu\,G)\sin^2 2\varphi} \qquad (9)$$

$$G' = \frac{E_x E_y G}{G\,[E_x + E_y\,(1 + 2\,\mu)]\sin^2 2\varphi + E_xE_y \cos^2 2\varphi} \qquad (10)$$

$$\eta_x = \frac{4\,G\,(E_x \sin^2\varphi - E_y \cos^2\varphi)\sin 2\varphi + E_y\,(E_x - 2\mu\,G)}{4\,G\,(E_x \sin^4\varphi + E_y \cos^4\varphi) + E_y\,(E_x - 2\,\mu\,G)\sin^2 2\varphi} \qquad (11)$$

$$\eta_y' = \frac{4\,G\,(E_x \cos^2\varphi - E_y \sin^2\varphi)\sin 2\varphi - E_y\,(E_x - 2\,\mu\,G)}{4\,G\,(E_x \cos^4\varphi + E_y \sin^4\varphi) + E_y\,(E_x - 2\,\mu\,G)\sin^2 2\varphi} \qquad (12)$$

The relations between the stresses related to the arbitrary coordinate system and stresses related to the coordinate system inclined at angle φ for isotropic as well as anisotropic bodies are the following:

$$\sigma_x' = \sigma_x \cos^2\varphi + \sigma_y \sin^2\varphi + \tau \sin 2\varphi \qquad (13)$$

$$\sigma_y' = \sigma_x \sin^2\varphi + \sigma_y \cos^2\varphi - \tau \sin 2\varphi \qquad (14)$$

$$\tau \;\; = \tfrac{1}{2}(\sigma_y - \sigma_x)\sin 2\varphi + \tau \cos 2\varphi \qquad (15)$$

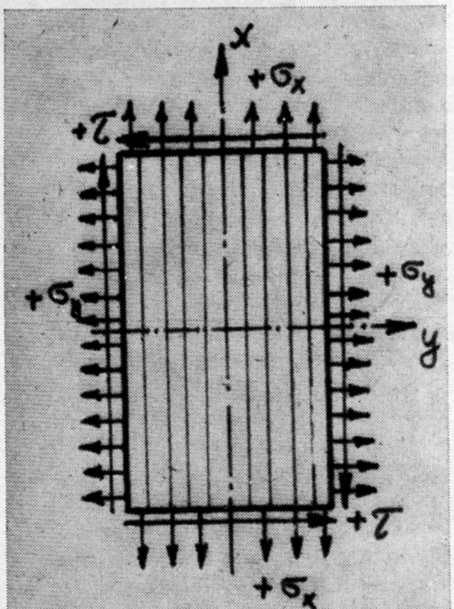


Fig. 1.

Elastic Properties of Multilayer Plate

Let us consider a plate consisting of several orthotropic layers of similar elastic properties (fig. 3). The layers are arranged in such a manner that their main elastic axes are inclined at angle 2φ. The sums of layer thickness in both directions are the same.

Except for identity of elastic modulus E_x, E_y, G and μ no other assumptions have been made. There may be as well the layers of parallel fibres bonded with artificial resin, as the layers of arbitrary structure (e.g. of component layers) but orthotropic. The particular layers may be of different thickness.
A plate of such construction is, of course, an orthotropic material with main elastic axes directed toward the

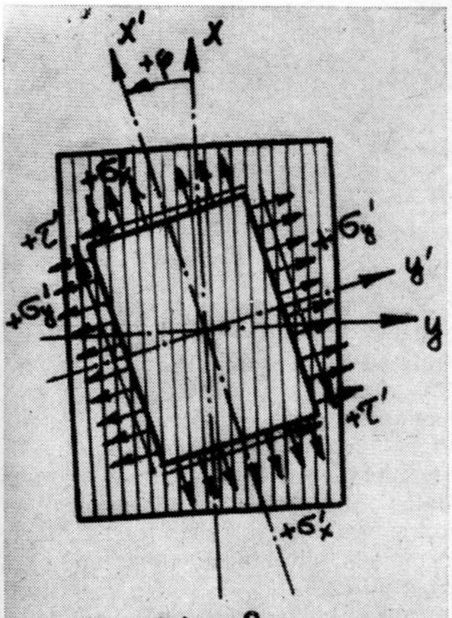


Fig. 2.

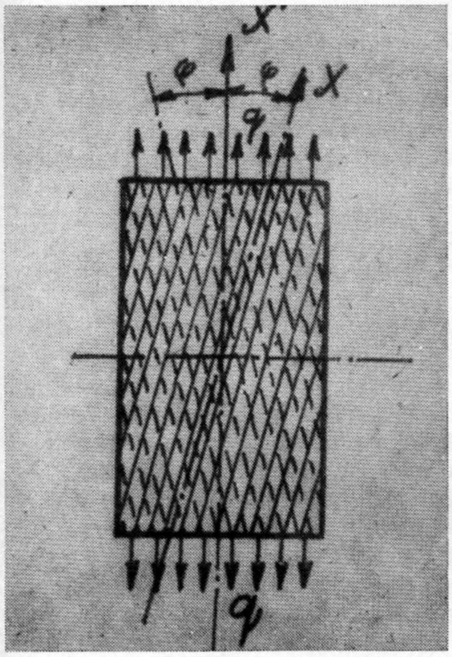


Fig. 3

bisector of angle 2φ. We search for the elastic coefficients of the plate E_{xp}, E_{yp}, μ_p and G_p with respect to plate main elastic axes assuming that the elastic coefficients of the layers E_x, E_y, μ and G with respect to the layer main elastic axes are known.
Let us consider the strain of the plate caused by external loading parallel to x-axis uniformly distributed along the plate edge and thickness, of value 'q' per unit of area.

If in consequence of the above loading the layers would be disjointed there would appear in them the strains ε_x', ε_y' and γ'. The strains ε_x' and ε_y' would be equal as to direction and magnitude for all the layers. The shear strain γ' would be equal in magnitude but of opposite direction for layers of different direction of main elastic axes.
The bonding (glueing) of layers manifests itself by involving the shearing stresses on the periphery of the plate which reduce the shearing strains to zero. These stresses, being due to the influence of one plate on the other, are arranged in opposite directions for the layers of different directions of axes, and provoke the appearance of linear strains along the axes x' and y'.
Since there the strains for all plate layers are equal as to magnitude and direction, they do not produce the interacting influence between layers, resulting only from the change of elastic coefficient of the plate when compared with the layer.
The interacting influence of the plates will be generally limited to a very narrow strip on the plate edge or in the vicinity of the region at which the load is applied.
There can exist a considerable stress concentration if the layers are of greater thickness. In regions where the stresses are constant there exists no interacting influence and in glue-joints the stresses are absent.
On the basis of the above considerations, by means of simple calculations, we obtain:

$$E_{xp} = \frac{E_x E_y \cos^2 2\varphi + G\,[E_x + E_y\,(1 + 2\,\mu)]\sin^2 2\varphi}{E_x \sin^4\varphi + E_y \cos^4\varphi + [G\,(1 - \frac{E_y}{E_x}\mu^2) + \frac{1}{2} E_y\,\mu]\sin^2 2\varphi} \tag{16}$$

$$E_{yp} = \frac{E_x E_y \cos^2 2\varphi + G\,[E_x + E_y\,(1 + 2\,\mu)]\sin^2 2\varphi}{E_x \cos^4\varphi + E_y \sin^4\varphi + [G\,(1 - \frac{E_y}{E_x}\mu^2) + \frac{1}{2} E_y\,\mu]\sin^2 2\varphi} \tag{17}$$

$$\mu_p = \frac{E_y\,\mu\,(\sin^4\varphi + \cos^4\varphi) - [G\,(1 - \frac{E_y}{E_x}\mu^2) - \frac{1}{4} E_x - \frac{1}{4} E_y]\sin^2 2\varphi}{E_x \sin^4\varphi + E_y \cos^4\varphi + [G\,(1 - \frac{E_y}{E_x}\mu^2) + \frac{1}{2} E_y\,\mu]\sin^2 2\varphi} \tag{18}$$

$$G_p = G\cos^2 2\varphi + \frac{E_x + E_y(1 - 2\,\mu)}{4\,(1 - \frac{E_y}{E_x}\mu^2)}\sin^2 2\varphi \tag{19}$$

For layers crossed perpendicularly (φ=45°):

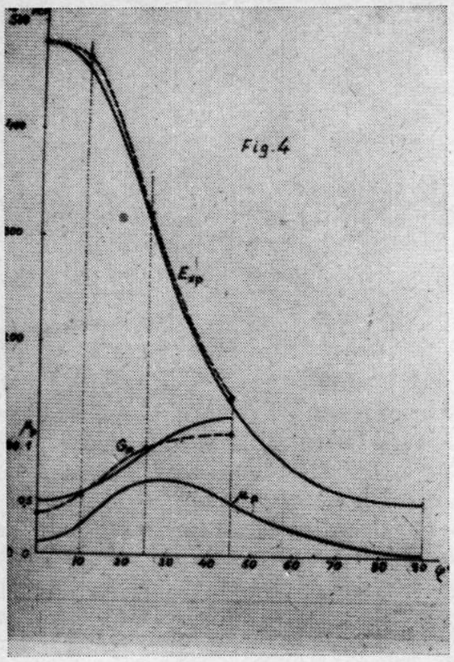
$$G_{p\,45°} = \frac{E_x + E_y\,(1-2\,\mu)}{4\,(1-\frac{E_y}{E_x}\mu^2)} \tag{20}$$

Arranging axes x and y in such a manner that: $E_x \geqslant E_y > 0$ and bearing in mind the orthotropic material relation $0 \leqslant \mu \leqslant 1$, it is easily shown that:

$$\frac{E_x + E_y}{4} \geqslant G_{p\,45°} \geqslant \frac{E_x}{4} \geqslant \frac{E_y}{4} \tag{21}$$

Comparison of Calculated Elastic Coefficients with Test Results

The test of fibre-glass/resin material properties have been carried on under the aegis of the Chair for Aeroplane Design in collaboration with the Chair for Strength of Materials and Structures, in the Technical University Warsaw.
The measurements concern the elastic coefficients E_{xp} and G_p for layers inclined at angles 2φ of 0° (all fibres parallel), 20, 50 and 90°. The measurements have been made on tube specimens of internal diameter of 20 mm with wall of 2 mm thickness. The specimens have been prepared by winding the roving Vetrotex RS 10–400–0–60 saturated with epoxy resin Epidian 53 with addition of 5% of styrene and 12% of hardener Z 1 (triethanoltetramine). The volume/glass ratio was about 55%. The wall consisted of four layers (two layers having the fibres inclined on angle φ on each side). When the hardening process had been completed



the wooden cylinder (on which roving had been wound) was pressed out. The measurement of E_{xp} was made by means of axial tension and compression and the stretch of a 20 mm gauge length was recorded. The measurement of G_p was performed by means of twisting the specimens and recording the twist angle on a gauge length of 150 mm.
Three specimens were used for each fibre inclination. For each specimen the measurements were repeated to obtain the average value for each of 27 measurements due to various loadings.

It has been stated that the elastic coefficients are constant over the range of applied loadings. The measurements of the elastic coefficients for tension, compression and shearing were performed on the same specimens. The results are given in table 1, in which is also included the standard deviation:

$$s = \sqrt{\frac{\Sigma\,(E - E_m)^2}{3-1}} \cdot \frac{100}{E_m}\,\%.$$

The average results of the elastic coefficient measurements are shown in figure 4 (. . . . .). To find the calculated values according to equations 16, 17, 18 and 19 it is necessary to know the value of four parameters: E_x, E_y, G and μ of the basic material with unidirectionally arranged fibres. This is the case of the specimen with fibres inclined at an angle φ=0°. On the basis of the measurements the true value, however, is obtained only for E_x because during the pressing out of the wooden cylinder there appear longitudinal cracks which considerably reduce the value of the shearing modulus G. (When the fibres are parallel to the direction of the shearing stresses, the stresses flow through the filling resin. When the fibres are inclined to the shearing stress direction the stresses are taken by fibres in tension or compression. On specimens with inclined fibres the detection of the cracks is rather difficult.) The coefficient E_y (corresponding to

Nr of specimen	$\varphi°$	$E_R\ 10^{-3}$ kG/cm²	$E_{Rm} \cdot 10^{-3}$ kG/cm²	s per cent	$E_c \cdot 10^{-3}$ kG/cm²	$E_{cm} \cdot 10^{-3}$ kG/cm²	s per cent	$G\ 10^{-3}$ kG/cm²	$G_m \cdot 10^3$ kG/cm²	s per cent
1	0	455			470			(14,2)		
2	0	537	481	10,2	506	481	4,4	35	36,5	–
3	0	450			468			40		
4	10	483			486			64		
5	10	461	467	3,1	452	468	3,7	54	55,7	13,7
6	10	456			467			49		
7	25	318			280			105		
8	25	344	321	6,9	328	290	11,5	105	100,3	8,1
9	25	300			263			91		
10	45	152			163			117		
11	45	145	148	2,4	153	157	3,3	114	114,3	2,2
12	45	148			156			112		

Tab. 1. Measured elastic coefficients of glass-fibre / resin specimen

$\varphi°$	$E_{xp}\ 10^{-3}$ kG/cm²			$G_p\ 10^{-3}$ kG/cm²			μ
	measured	computed	discord per cent	measured	computed	discord per cent	
0	481	481	0	36,5	47,0	22,3	0,1
10	467	455	2,6	55,7	56,8	1,9	0,278
25	321	320	0,3	100,3	96,0	4,5	0,69
45	148	140	5,7	114,3	130,6	12,5	0,454
60		81					0,216
90		50					0,01

Tab. 2. - Measured and computed elastic coefficients of glass-fibre/resin specimen

φ=90°) and μ were not measured. The influence of coefficient μ on results obtained from equations 16, 17, 18 and 19 is rather poorly defined and μ changes in quite a narrow region. So its value has been taken from the literature, as μ=0.1. The corresponding values of E_y and G have been taken from measurement results for specimens with fibres inclined at an angle $\varphi = 25°$.
From the above and using equations 16 and 19 it is found that the value of E_y=50,000 kG/cm² and of G=47,000 kG/cm² approximately. (This is the reason for some differences between the computed and measured values of E_{xp}, G_p for φ=25°.)
The results of the calculations are presented in table 2 and figure 4 (———).
The agreement between calculation and measurement results is satisfactory. Per-cent differences are included in table 2.
The separate calculation of E_{yp} and G_p for φ>45° is unnecessary in view of the obvious relations:

$$E_{yp\varphi} = E_{xp\,(90°-\varphi)}$$
$$G_{p\varphi} = G_{p\,(90°-\varphi)}$$

The measured values can differ from the above results because the specimens were of cylindrical shape, and the curvature has some influence on the value of the coefficient for a given angle of inclination to the generating line (the low curvature of fibres for φ<45°) as well as affecting the precise location of the tangent to the periphery of the pipe, on account of the great curvature of the fibres.
The above may be the reason for the differences between the measured (on pipe specimens) and calculated (for plane plate) coefficients.

References

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