Numerical computation of lee wave flow including rotors

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Paper presented at the XIth OSTIV Congress 1968, Leszno, Poland

Summary
A method is described of using a high speed computer to plot flow patterns and orographic cloud over and in lee of high ground. Care is needed in making boundary assumptions and in interpreting the results, but the facility can be a powerful aid to observational studies of mountain airflow including lee waves, rotors and possibly local blocking and high level turbulence.

1. Introduction

In two papers in 1949 and 1953 Scorer [1, 2] discussed the occurrence of lee waves in the atmosphere and showed how the values of possible lee wave lengths may be calculated for an airstream that can be considered as two or more layers each with a constant value of a parameter depending on the wind and temperature structure. The calculations become very lengthy if the number of layers exceeds three, so early calculations were restricted to simplified airstreams with only a few distinct layers. Corby and Wallington [3] showed that in some situations lee wave amplitudes are very sensitive to small changes in the airstream, and, although simplified models may be adequate in many situations, there is no simple criterion for assessing whether such simplification is justifiable for any particular airstream. Therefore, when calculating lee wave amplitudes it is wiser to overcome the computing problem than to apply the theoretical technique to a crude approximation of a real airstream.

Use of a high speed computer for numerical studies of lee waves was described by Wallington and Portnall [4, 5] in 1958. This work was exploratory and the program was not designed for a general purpose numerical study of the lee wave features of mountain airflow. However, with a modern computing system it is possible to compute and present some mountain airflow calculations in such a way that they can be used not only to study lee waves but also as an adjunct to studies of rotor flow, local blocking by mountain ridges and high level turbulence.

2. Formulation of the problem

In this paper we shall consider only small perturbations in frictionless, steady, isentropic flow in two dimensions in a vertical plane and limit the discussion to waves that are short enough for the coriolis force to be neglected. If ψ is the perturbation displacement of a streamline from its original level, z, then the gas equation, the adiabatic equation and the equations of motion and continuity can be arranged to show approximately that

$$\frac{\partial^2 \Psi}{\partial z^2} + \left(\frac{2}{U}\frac{\partial U}{\partial z} - \frac{1}{\theta}\frac{\partial \theta}{\partial z}\right)\frac{\partial \Psi}{z} + \left(\frac{g}{\theta U^2}\ \frac{\partial \theta}{\partial z}\right)\Psi = 0$$

where x denotes distances along the horizontal axis of the vertical cross-section,

U is the undisturbed component of the wind in the x direction at height z, and Θ is the undisturbed component of the potential temperature at height

In order to seek solutions which are sinusoidal in the x direction we substitute ψ = the real part of $f(z)e^{ikx}$ where f(z) is a function of z only, k represents wave numbers to be determined and $i = \sqrt{-1}$. Eq. (1) may then be written

$$\frac{\partial^2 f(z)}{\partial z^2} + 2S\,\frac{\partial f(z)}{\partial z} + (l^2 - k^2) f(z) = 0$$

where S and l are functions of z only.

If we follow Scorer and consider the flow across a mountain ridge whose height, h, is given by

$$h = \frac{Hb^2}{b^2 + x^2}$$

where H is the height of the crest, and b is a width parameter, then the solution for the streamline displacement becomes

$$\Psi = \text{real part of } Hb \int_{k=0}^{k=\infty} e^{-k(b \pm ix)}\, \frac{f(z)}{f(0)}\, dk$$

where f(o) denotes the value of this function at ground level, z = o.

The integral will have singularities for any values of k that yield f(o) = o. These values of k(= K, say) are the lee wave numbers and the flow patterns corresponding to them are called the lee waves.

The integral has an ambiguity of sign ($\pm$ix) and several authors, including Queney [6], Palm [7] and Corby and Sawyer [8], have tried to resolve this ambiguity but there is no generally accepted proof that one sign is more appropriate than the other. However, in the situations that will be discussed here the effect of the ix is negligible.

It can be shown that for ridges that are not narrow (say b not less than about 1 km), the solution represented by Eq. (4) is approximately

$$\psi = \frac{Hb^2}{b^2 + x^2}\,(f(z)/f(0))_{k=0} - 2\pi H b e^{-kb}\left[\frac{f(z)}{\left(\frac{\partial f(0)}{\partial k}\right)}\right] \sin Kx$$

where the second term on the right applies only to the lee side of the crest of the ridge and is repeated for all wave numbers K than can exist in the flow.

As Eq. (1) is linear the flow pattern across any high ground profile can be determined by resolving the cross-section into a number of ridges of the type specified by Eq.(3), computing the flow across each of these ridges, and adding the flows together to form the complete pattern across the high ground cross-section. In the Computing Research Division of CSIRO, Australia, a computing program has been written to calculate flow patterns in this way, and to compute, as by-products, wave flow parameters such as the l in Eq. (2) and the effects of the wave flow on radio-sondes ascending through the disturbed airstream.

3. The Computing Program

Data required for the program comprises

I. Airstream data:

Winds, temperatures and dewpoints at as many levels as are available. The levels at which these items are given can be in millibars, feet, or metres; the program deals with any of these units. Wind directions are in degrees; speeds are km/h, m/sec, mi/h or knots. Temperatures are °K, °C or °F and dewpoints are not essential if cloud computations are not required.

II. High ground data:

A list of heights at specified regular intervals on a cross-section of the high ground in a specified direction. Alternatively, if an experimenter wishes to study the flow over one or more ridges of the type specified by Eq. (3) he need only give the height, width and position parameters of the ridges. The program accepts up to 250 heights or 100 sets of ridge parameters.

III. Boundary levels and number of

The heights of the upper and lower boundaries between which the airflow computing levels:

calculations will be made and the number of levels between these boundaries that will be used for finite differences in the vertical finite difference form of Eq. (2). Regarding boundary conditions, see Section IV below.

IV. Release points and rates of ascent of simulated radio-sondes:

If the experimenter wants the program to calculate the temperatures, dewpoints and wind speeds that would be measured by a radio sonde ascent through the flow, he must specify the release point of the radio-sonde and the rate of ascent of the sonde in ft/min, ft/sec, m/sec or km/h.

With this data the program goes through all or a selection of the following actions:

a) The airstream data is printed out and plotted in graphical form on a Calcomp plotter. If dewpoints are concluded in the data the plot indicates by hatching any layers where the dewpoint is within 1° C of the temperature – this 1° C separation being an arbitrary but convenient indicator of probable cloud. Fig. 1 shows a typical plot.

b) Temperatures, dewpoints and wind components across the high ground for the levels to be used in vertical finite differences are calculated by interpolation from the airstream data. If the upper boundary is above the highest level at which temperature or wind data are available then the program assumes that the value is constant with height above this level.

c) The parameters S and l for Eq. (2) are computed and graphs of l and $2\pi/l$ are plotted against height. The physical interpretation of the 'natural wavelength', $2\pi/l$, was described by Wallington [9] in 1958. Fig. 2 shows graphs for the data illustrated in fig. 1.

d) Starting with values of l at the top z levels the simplest finite difference form of Eq. (2) is integrated down to the lower boundary with k = o. Thus values of $[f(z)/f(o)]_{k=o}$ are available for the approximate displacements given by the first term on the right in Eq. (5).

e) Starting with similar boundary conditions to those just described the finite difference form of Eq. (2) is integrated down to the lower boundary repeatedly for values of $k = \varepsilon, 2\varepsilon, 3\varepsilon, --- n\varepsilon$, in a search for wave numbers, K, that would yield f(o) = o. Wherever the computed values of f(o) for a pair of successive values of k change sign a wave number, K is taken as the mean of these two values of k, and the pairs of values f(o) and k are used to compute $(\delta f(o)/\delta k)_{k=K}$. Trials have shown that $\varepsilon = 0.003\ km^{-1}$ is a practical value for adequate resolution of wave numbers, while n = 1200 allows the search to extend up to 3.6 km^{-1} (i. e. lee wavelength down to approximately 2 km).

f) Graphs of the barostromatic displacement factor and lee wave amplitude factor are plotted against height on the Calcomp plotter. Fig. 3 shows graphs for the data in fig. 1.

g) The high ground profile is analysed into ridges of the form specified by Eq. (3). The method of doing this is to search for the highest crest in the profile. This is taken as the crest of a ridge of height, H, and the associated value of the width parameter, b, is taken as half the distance between the two points nearest to, and either side of, the crest where the height equals ½ H. These H and b values together with the position of the ridge are taken as the main elementary ridge in the profile. Its height is computed for each point and subtracted from the given heights. A search is made for the highest crest in the residual heights to find the next most significant ridge in the profile and the process is repeated until the residual heights are negligible. For many high ground profiles the number of elementary ridges needed to account for all the significant features is usually less than one fifth of the number of given points in the cross-section. The program analyses n given heights into n/5 ridges.

h) The heights of 20 streamlines between the upper and lower boundaries are calculated such that they represent the undisturbed wind components, i. e. the wind speed is inversely proportional to the streamline spacing.

i) The following calculations are made for each streamline. For each point over the given ground cross-section a search is made from ground level upwards to find the height at which the

Fig. 1
Profiles of temperature, dewpoint and wind component from 270 deg. for 0950 EST 17 April, 1966 at Hobart, Tasmania. The shaded layer is a cloud (assumed likely if the dewpoint is within 1° C of the temperature).

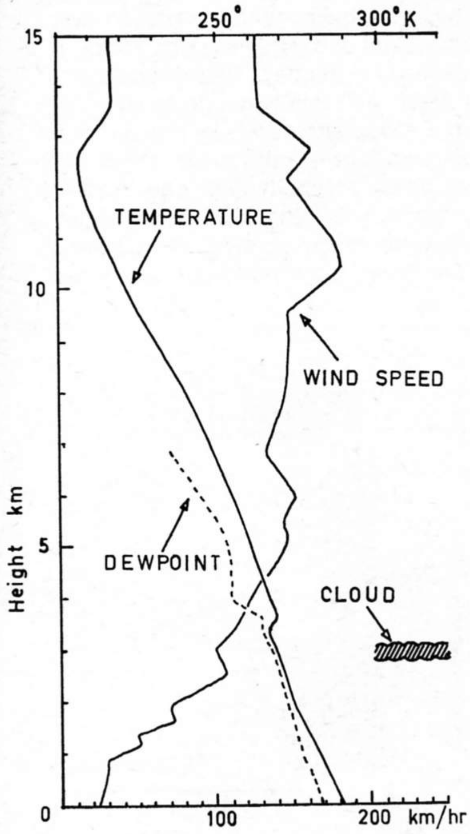


Fig. 2
Computer plotted profiles of «l» and the «natural wavelength» for the data illustrated in fig. 1.

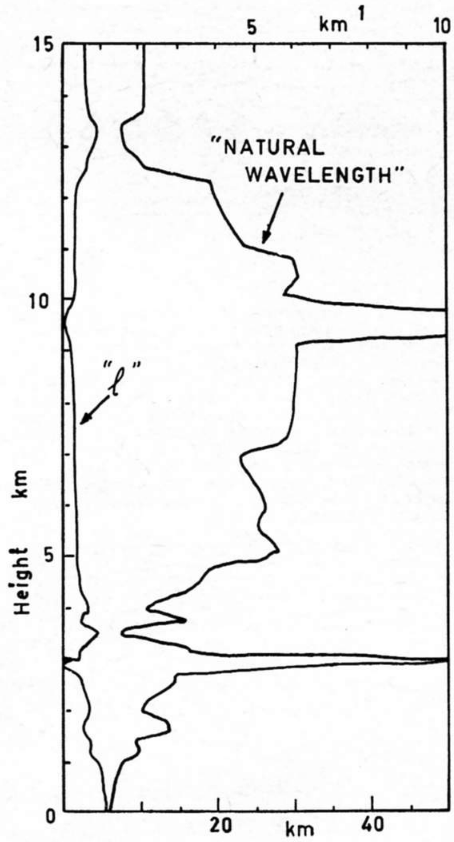
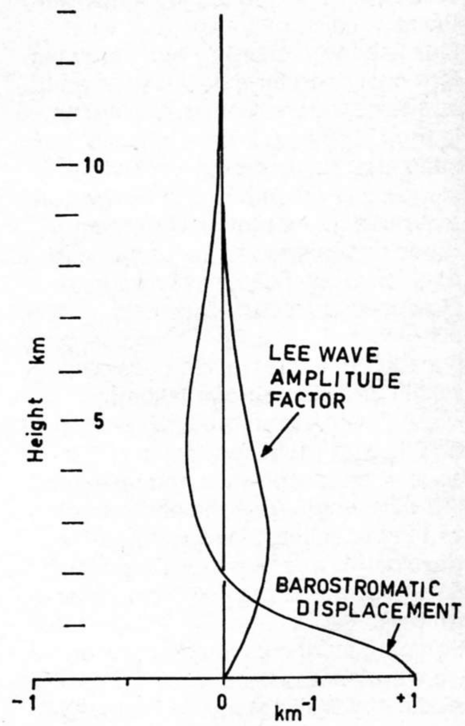


Fig. 3
Computed profiles of barostromatic displacement and lee wave amplitude factors for the data illustrated in fig. 1. The computed lee wavelength was 18.5 km. In this case conditions at high levels appeared to have little or no effect on the flow below about 10 km.



streamline should be to make its displacement equal to the sum of the barostromatic displacement and the displacements due to lee waves from all upstream ridges. The streamline is drawn on the plotter in a vertical cross-section showing the flow over the high ground. Over some points on the high ground profile there may be more than one possible displacement for a streamline. This means that there is a local reversal or vertical circulation in the flow. The computing program can deal with three possible streamline displacements and draw a closed circulation, but it does not search for more than three possible displacements. More complicated flow patterns appear to be too rare to justify introducing more complications into the computing program.

j) If the airstream data includes dewpoints the program computes and draws the cloud pattern in the airflow on the assumptions that cloud forms if the temperature is within 1° C of the dewpoint and that descending air in a layer of cloud does not become unsaturated until it descends below the base of the layer.

k) If the data includes release points and rates of ascent of simulated radio-sondes, the paths of the sondes are calculated and drawn on the cross-section, graphs of the temperatures, dewpoints and wind speeds that the sonde would measure are plotted, and the vertical speeds of the air through which each radio-sonde ascends are plotted for each level used in the computations.

l) A pressure scale is drawn alongside the cross-section to allow altitude to be read as a pressure if required, and wind scales are drawn to allow an experimenter to measure horizontal and vertical wind speed components from the streamline pattern.

Fig. 4 shows the airflow and cloud pattern computed for a flow whose wind and temperature profiles are illustrated in fig. 1. The clouds were shaded in by hand after the computation but the cloud outlines were drawn by the computer controlled plotter. Airstream winds and temperatures for this computation were those measured by radio-sonde at Hobart, Tasmania, at 0900 EST 17 April, 1966. This airstream produced bars of lee wave clouds clearly visible on a photograph taken by ESSA 2 satellite on orbit 600 at 0740 EST 17 April, 1966. Andersen [10] described the occurrence and measured the wavelength from the photograph as 11 naut. miles. The computed lee wavelength was 10 naut. miles, which is in good agreement with the observations.

Comparison of computed an observed lee wavelengths were made for cases studied by Wallington and Portnall [4] suggested that the computing procedure was satisfactory for many situations.

4. The boundary assumptions

The experimenter can choose the upper and lower boundary levels of the computed flow pattern. The lower boundary is taken to be either ground level or some surface which acts as an effective lower boundary, e. g. if winds are practically calm in a shallow layer of stable air below, say, a few hundred metres. In this case the experimenter may wish to try a calculation on the assumption that the lower boundary is virtually the top of this almost stagnant air together with whatever high ground protrudes up through it. In some computations streamline displacement is found to be zero at a level below the top of the high ground in the region. In such a situation it must be assumed either that the theory is inapplicable or that the high ground will block or divert the low level flow. If a level of zero displacement is below the crests of a mountain range but above the level of some of the passes, it may be argued that the airflow will be concentrated as particularly strong winds through these passes. But this is conjecture at present. The main points being made here is that in some situations the choice of a low level boundary is a subject for experimental and observational studies, and that the computing facility is an aid to such studies.

In setting the upper boundary conditions the experimenter can set the height and the value of f(z) at the top level. By setting this value, f(z) he sets the ratio of the values at the top two levels; the magnitudes of the displacement values are scaled by the lower boundary factors in the displacement terms. Wallington and Portnall have discussed the effect of the upper boundary assumptions on lee wave calculations in detail; in general, if the displacement curves tend to small values towards the upper boundary, or if variations in upper boundary assumptions do not significantly change the flow pattern at lower levels, then it is likely that the flow pattern computation is valid.

5. Rotors

Because the displacement calculated for a level z refers to displacement at that level and not from that level it is possible for a streamline to have more than one level over any point on the ground profile. If a streamline has three possible positions it must have either an S shape or be part of a closed circulation in the vertical cross-section. The computing program includes a facility to detect and draw such features. Fig. 5 shows a computed pattern with such features. When this situation actually occurred turbulence below about 2 km was so violent that light aircraft flying was abandoned for the day. Turbulence associated with this type of rotor flow is not produced merely by the vertical circulation; it is the result of the smaller scale instability that is generated when overturning of air in the stable layer forces warm air under cooler air. Fig. 6a shows a well developed wave

Fig. 4
The computer plotted streamline and cloud pattern for the airstream represented by the data in fig. 1 flowing over a high ground West–East cross-section through Hobart. The clouds were shaded in by hand but their outline was drawn by the computer plotter. The calculated lee wavelength of 18.5 km (10 naut. miles) is in good agreement with a wavelength of 10 naut. miles observed by satellite photography.

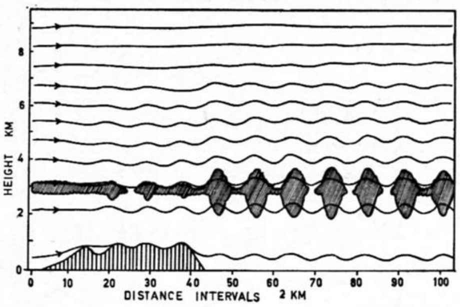


Fig. 5
Computed flow pattern over a West–East cross-section of the Brindabella Mountains, Australia.
The airstream wind and temperature data was that measured by radio-sonde at 09 EST 12 July, 1966 at Wagga Wagga, New South Wales. If streamlines had been drawn at closer intervals they would have shown additional closed circulations, or rotors, under the two wave crests near the centre of the pattern. Turbulence encountered below about 2 km on this day was so violent that light aircraft flying was abandoned for the day. The airstream contained a shallow, 5° C temperature inversion between 900 mbs and 890 mbs.

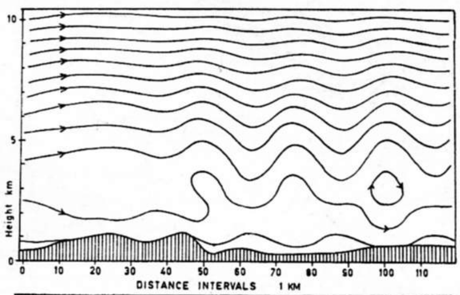
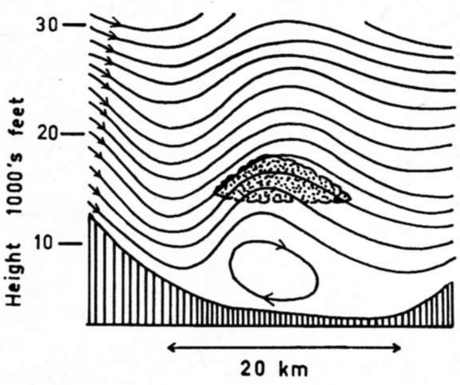


Fig. 6a
The flow pattern deduced from observations and measurements on 16 February, 1952. The observed lee wavelength was 18 km at about the 3 km level.



and rotor flow investigated during the Sierra Wave project. This has been described by Holmboe and Klieforth [11]. Fig. 6b shows the computed flow pattern for the situation. The airstream data used was that from Merced at 1200 P (2100 GMT) 16 February, 1952. Both the observed and the computed lee wavelengths were 18 km, and although the plotted streamlines do not happen to include a closed isopleth, rotor flow circulation below the wave crests was evident from more detailed printed results.
Some computed flow pattern have rotor flow features at high levels. These may well be indicators of high level turbulence, the computing technique may well be an aid in investigations of such turbulence, especially as it is favoured rather than inhibited by static stability. But in the computations made so far there is not enough evidence to assert whether the high level rotors are likely to be real or spurious byproducts of upper boundary assumptions.

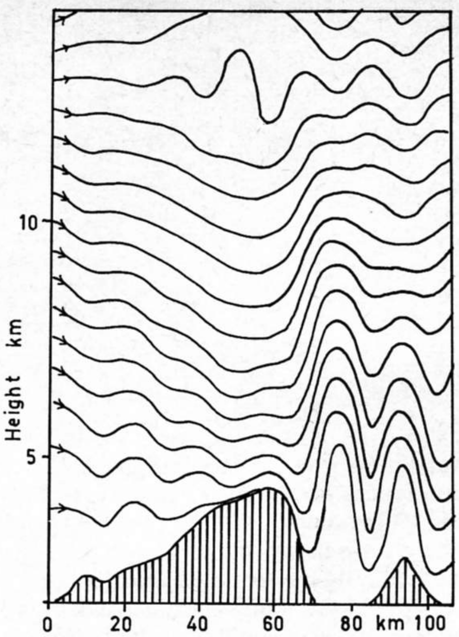


Fig. 6b
The computed flow pattern for 16 February, 1952. Airstream data used was that obtained at Merced at 1300 P (2100 GMT) 16 February, 1952.
The computed streamlines do not include an isopleth to outline a rotor circulation, but such circulation beneath the wave crest was evident in more detailed printed results. The orographic cloud pattern is omitted for clarity; it contained wave clouds in the correct locations with tops at 5 km, but the bases were much too low.

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