The Vertical Velocity Pattern Created by an Isolated Mountain

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Summary of paper presented at the 6th OSTIV Congress, St-Yan (France), July 1956

The purpose of this article is to obtain an analytic expression for the vertical velocity pattern in three dimensions in the lee of an isolated mountain, and to present this in a form such that the comparison with the original two-dimensional study of Lyra [1] will be facilitated. Some numerical solutions for the vertical velocities have been obtained by R. S. Scorer [2] for a mountain of a shape such that there is non-oscillatory disturbance to the lee, and these results may also be used for comparison. The computed pattern turns out to be one characteristically observed in the lee of an isolated hill or mountain range of limited length, and cloud photographs are adduced in this condition.

The model considered, identical with that of Lyra, consists of a stably stratified atmosphere flowing with uniform speed along the x-direction. It is assumed that the space-time scale of the disturbance renders negligible both the compressibility of the air and the coriolis deflection. The disturbed motions are thus pure gravity waves with non-negligible vertical accelerations. The disturbance introduced by the mountain is assumed to be small of the first order. The equation governing the vertical velocity w in such a system is

$$\frac{\delta^2}{\delta x^2}\left(\frac{\delta^2 w}{\delta x^2}+\frac{\delta^2 w}{\delta y^2}+\frac{\delta^2 w}{\delta z^2}\right)+k_s^2\left(\frac{\delta^2 w}{\delta x^2}+\frac{\delta^2 w}{\delta y^2}\right)=0 \qquad (1)$$

where $\quad k_s^2 = g\left(\gamma_d - \gamma\right) \Big/ U^2\, T$

g being the acceleration of gravity $(\gamma_d - \gamma)$, the difference between dry-adiabatic and environmental lapse rates, U the current speed, and T the mean tropospheric temperature. This equation simplifies to that of Lyra [1] and Queney [3] in case there is no variation along the y-direction.

The fundamental solution of Lyra was for an infinite plateau of height h. In the present model the plateau will have width $2\,b$ in the y-direction. If b is 1 kilometer or less, the following solution applies. The steady-state formulation of the problem, as is well known, is indeterminate, and admits of a symmetric solution, when what is wanted is a solution with a lee wave but negligible disturbance at large distances upstream. This aspect of the problem has been discussed in the literature (Hoiland [4]; Palm [5]; Wurtele [6], so that for present purposes it will be sufficient to use the method of Queney [3, pp. 41 ff.] for eliminating mathematically the upstream wave.

It would be possible to obtain a numerical solution to equation (1), but for the purposes at hand an asymptotic evaluation is preferable. In the two-dimensional case the asymptotic solution is a good approximation at distances greater than one stationary wave length (abouth 7 km) from the mountain; closer than this in nature, the particular shape of the mountain plays an important role, and the non-linear effects probably dominate the motion. The desired solution is

$$w = 2k_s h\, b\, U A \cos\left(k_s R z / r\right) \qquad (2)$$

where

$$R^2 = x^2 + y^2 + z^2 \;,\; r^2 = x^2 + z^2,$$

and

$$A = \frac{x z \left(r^4 + x^2 y^2\right)^{1/2}}{r^3 R}\left\{1 + 4\left(\frac{x y z R}{r^4 + x^2 y^2}\right)^2\right\}^{1/2}$$

The geometric description of the wave pattern in three dimensions is not complicated. In the plane $y = 0$ the solution (2) reduces to

$$w = 2\, k_s h\, b\, U \frac{x}{x^2 + z^2} \cos k_s \left(x^2 + z^2\right)^{1/2}$$

which is the wave pattern of Lyra's two-dimensional asymptotic solution for large r. Lyra's complete solution is reproduced in figure 1. The magnitude of the disturbance downstream, however, dies out like x^{-1}, as compared with $x^{-1/2}$ in the two-dimensional problem. This is the result of the spreading of the wave in the direction normal to the flow.

The tilt of the wave decreases with increasing distance from the x-axis. The lines joining the first downstream crests in several planes y = constant are displayed in figure 2. The line in the plane $y = 0$ thus corresponds to the first crest line in figure 1.

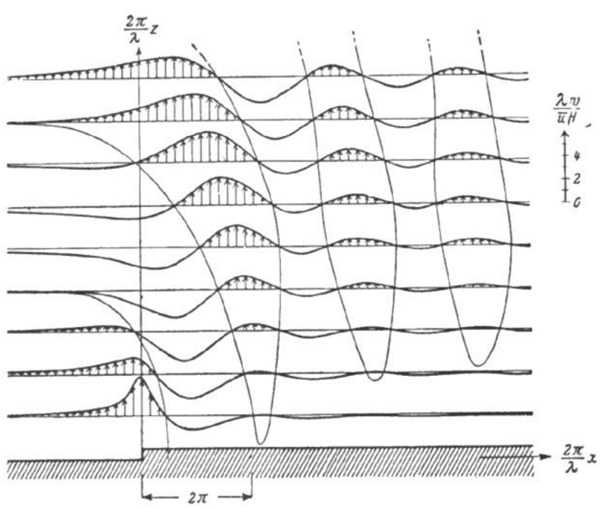

Fig. 1. Vertical velocities in the two-dimensional lee wave (after Lyra [1]). The distance of the heavy lines from the horizontal to the vertical velocity. Under the typical meteorological conditions described in the text, one non-dimensional vertical velocity unit is equal to 1.9 m/sec.

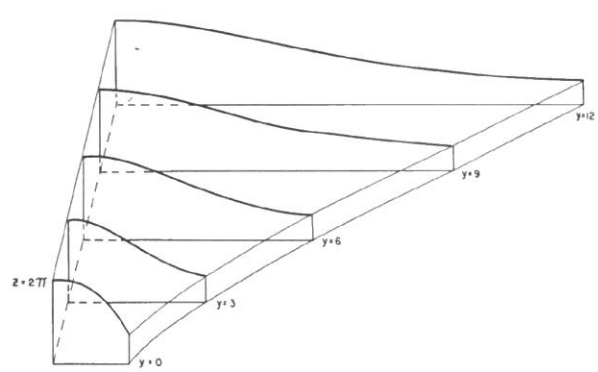

Fig. 2. The tilt of a wave crest as a function of cross-wind distance, shown in five planes, y = constant

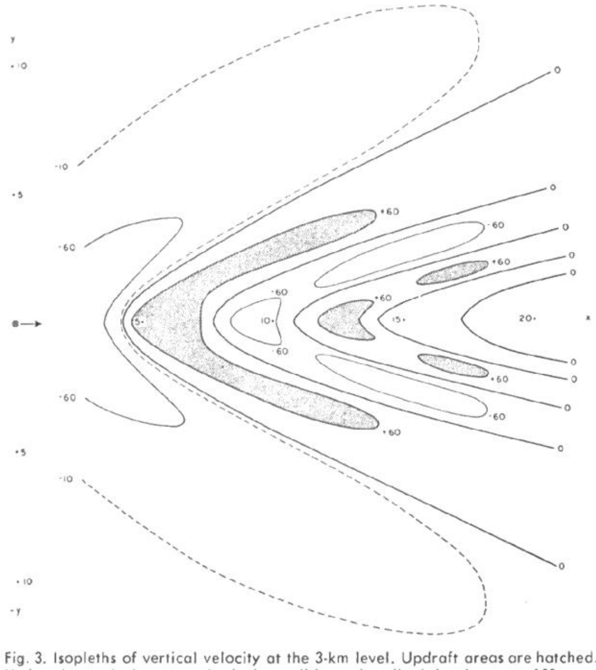

Fig. 3. Isopleths of vertical velocity at the 3-km level. Updraft areas are hatched. Under the typical meteorological conditions described in the text, 100 non-dimensional vertical velocity units are equal to 45.1 cm/sec.

In horizontal planes $z = z_0$ the nodal lines are given by (2). If we take the point $x = x_0$, $y = 0$ on any one of these lines, its equation becomes

$$\frac{x^2}{x_0^2} - \frac{y^2}{z_0^2} = 1$$

so that the nodal lines in horizontal plane are rectangular hyperbolas with the x-axis as their major axis, concave downwind.

When isopleths of vertical velocity are plotted, the factor A in (3) alters the shape, but the pattern is always dominated by the hyperbolic nodal lines. To gain an idea of the magnitude of the vertical velocities, we may take the height h of the plateau as one kilometer and the half-width b as one kilometer. The wind speed will have the value 20 meters per second, and the stationary wave length corresponding to this speed and to a lapse rate of about 65 % of the dry-adiabatic is $L_s = 10.5$ kilometers, $k_s = 0.6$ km^{-1}. The non-dimensional vertical velocities for the 3 kilometer level are analyzed in figure 3, and the dimensional values for the case mentioned are indicated.

The crescent shape of the first updraft areas (hatched in figure 3) is a frequently observed cloud pattern. The best photograph of such a cloud that I have been able to find is one taken by Count Masanao Abe [7] in the lee of Mount Fujiyama, reproduced in figure 4. As a matter of general meteorological interest, and also to call attention to the originality of Count Abe's researches, I include a photograph (figure 5) of Abe's model experimental wind tunnel, in which he successfully reproduced the crescent shaped lee cloud.

Figure 3 may also be compared with figure $3a$ of Scorer's article [2]. Although we have no analytic expression for the latter pattern, and although there would be no wave pattern if the computation were continued downstream, a simple visual comparison of the figures reveals a marked similarity in the crescent shaped updraft region. The difference in downstream wave patterns is a result of the particular form of Scorer's mountain and exists in two dimensions as well as in three (cf. Queney, op. cit.).

Fig. 4. Crescent shaped cloud in the lee of Mt. Fujiyama, photographed by M. Abe [7]

The only other similar gravity wave in three dimensions which has received attention in the literature is the ship wave. First studied by Lord Kelvin, this problem received perhaps its most complete treatment by Hogner [8]. Mathematically, the problem is one of a moving disturbance of the pressure at the free surface of a liquid of infinite depth. The resulting displacements are diagrammed in figure 6.

The most striking feature of the ship wave is the wake, with an angle of 39 degrees, 16 minutes. The wake lines are the waves with greatest elevation and outside these the disturbance is negligible. No such effect occurs in the atmospheric model, although, to be sure, the amplitude here falls off as y^{-5}. This cannot be explained in satisfactory detail without reference to the corresponding transient problem.

There is a further difference in the pattern of crests; the ship wave solution exhibits a singularity along the x-axis as a result of the occurrence of the impressed pressure disturbance on the free surface itself. In the atmospheric case, of course, the disturbance is initiated at $z = 0$, the entire motion on this plane being kinematically constrained.

The ship wave pattern might be expected to be characteristic of the lee wave on an atmospheric surface of discontinuity, which behaves dynamically like a free surface. This is in fact the case. Scorer [9] has recently employed such a model with an inversion at two kilometers, obtaining

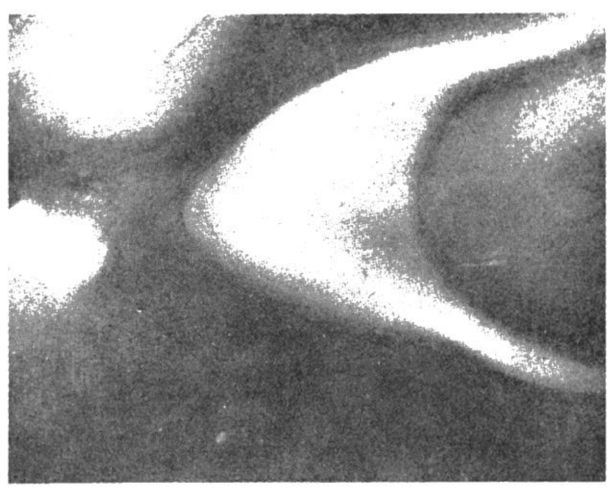

Fig. 5. Crescent shaped lee wave cloud formed in model experimental wind tunnel of M. Abe [7]

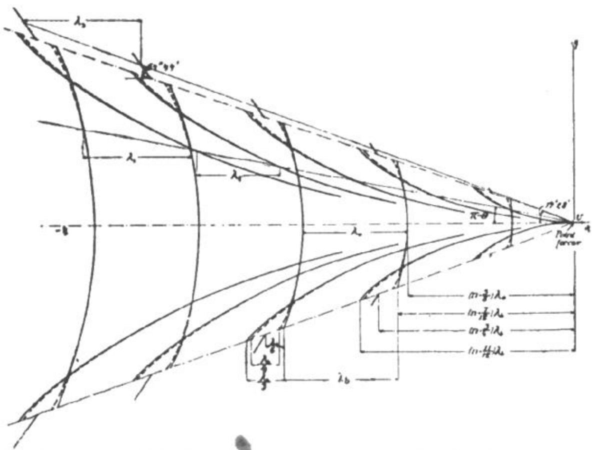

Fig. 6. Elevations of the free surface in a ship wave (after Hogner [8])

a wake angle of about 25°. In the upper layer, of course, the internal lee wave pattern discussed in this article would be superimposed on the ship wave.

For full mathematical details the reader is referred to [10].

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