An Analysis of Goodhart's Figure of Merit

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Introduction

H. C. N. Goodhart has proposed a Figure of Merit which describes the potential cross-country performance of a glider and is independent of the weight of the glider. It is obtained by taking the maximum value of the ratio (average cross-country speed) : (thermal strength).

The purpose of this paper is to provide a theoretical justification for this Figure of Merit, to obtain its value for an idealised glider and some real gliders and to consider the practical implications of the theory.

General Derivation of the Figure of Merit

Given the performance curve of a glider, there is a well-known construction for obtaining the best speed to fly between thermals and the average speed achieved, given the achieved rate of climb. This is shown graphically in Figure 1, where

v_c = achieved rate of climb
V = best speed to fly between thermals
v_s = corresponding rate of sink of the glider in still air
$\overline{V}$ = average cross-country speed.

For convenience, this diagram may be made non-dimensional by writing:

$$r = V/V_o,\ s = v_s/v_o,\ c = v_c/v_{so},\ u = \overline{V}/V_o$$

where

V_o = minimum drag speed, and
v_{so} = rate of sink at V_o.

For the sake of simplicity, true speeds will be used throughout this paper, it being implied that conditions near sea-level are under consideration.

Figure 2 then shows the require construction in a non-dimensional form. For consistency in signs, rates of sink are negative and rates of climb are positive. In this diagram, the performance curve or "polar" becomes independent of weight and is therefore a unique curve for each type of gliders in a given configuration.

Δ, the non-dimensional rate of sink, is therefore a unique funktion of r, the non-dimensional forward speed, for each type of glider.

i. e. $\qquad \Delta = S(r)$

where the nature of the function S depends on the type of glider.

With the sign convention shown above, it follows that at each chosen value of c, the non-dimensional rate of climb,

$$S' = \frac{-c+\Delta}{r} = \frac{-c+S(r)}{r}$$

i. e. $$c = S - rS' \qquad (1)$$

where S' denotes $\frac{d}{dr} S(r)$

also $$-\frac{c}{u} = S'$$

i. e. $$u = -\frac{c}{S'} = r - \frac{S}{S'} \qquad (2)$$

Now if the true thermal strength $= v_t$, then $v_t = v_c +$ (rate of sink of the glider in circling flight)

$$= v_c + \sigma\, v_{s\,\min}$$

where $v_{s\,\min}$ = minimum rate of sink in straight flight,
and σ = a constant depending on how the glider is flown in circling flight.

Now, for any given glider

$v_{s\,\min} = K\, v_{so}$ where K is a constant

so $v_t = v_c + \sigma K\, v_{so}$
$\qquad = v_c + n v_{so}$ say.

i. e. $$\frac{v_t}{v_{so}} = c + n \qquad (3)$$

Hence, given S and n and choosing values of r, u could be found from equation (2) and c from equation (1). u could then be plotted as a function of $(c + n)$ for the given glider (Fig. 3).

In order to obtain Goodhart's proposed Figure of Merit, the slope of the tangent from the origin is required, i. e. the maximum value of

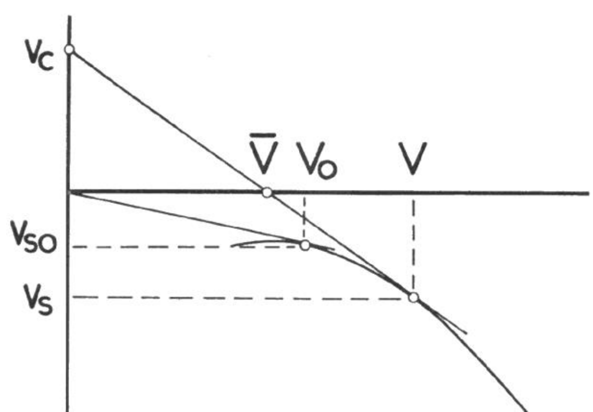

Fig. 1

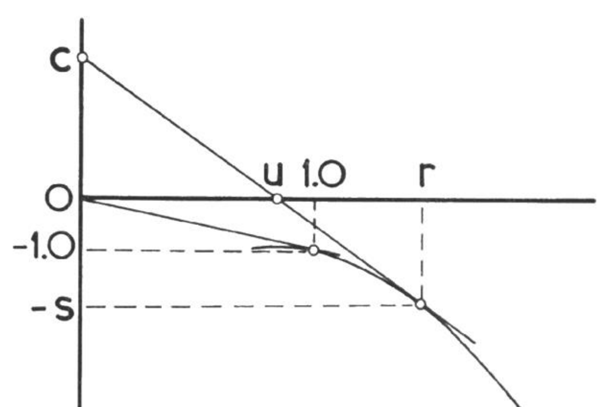

Fig. 2

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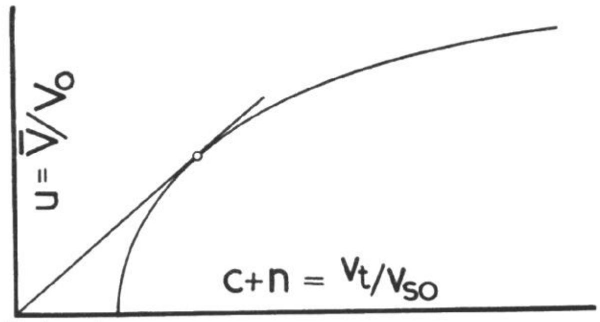

Fig. 3

$$\left(\frac{u}{c+n}\right)$$

i. e. of

$$\frac{rS'-S}{S'(S-rS'+n)}$$

By differentiation, it is found that the maximum value occurs at the value of r which satisfies

$$nS + (rS'-S)^2 = 0 \tag{4}$$

The value of $\left(\frac{u}{c+n}\right)_{\text{max.}}$ is clearly independent of weight and is unique for a given glider.

Goodhart's Figure of Merit is $\left(\frac{\overline{V}}{v_t}\right)_{\text{max.}}$

Now
$$\left(\frac{\overline{V}}{v_t}\right)_{\text{max.}} = \left(\frac{\overline{V}/V_o}{v_t/v_{so}}\right)_{\text{max.}} \times \left(\frac{V_o}{v_{so}}\right)$$
$$= \left(\frac{u}{c+n}\right)_{\text{max.}} \times \left(\frac{L}{D}\right)_{\text{max.}}$$

Since both of these factors are independent of weight, it follows that the Goodhart Figure of Merit depends only on the aerodynamic properties of the glider—on the *shape* of the performance curve as defined by S and on the best lift/drag ratio.

It is also clear that this Figure of Merit, which might conveniently be termed the "Goodhart Number", is a measure of the *potential* cross-country performance of a glider. Considering the actual curves of $\overline{V}$ against v_t, it will be seen that different values of the weight will give different curves, whose envelope is a line from the origin, of slope $(\overline{V}/v_t)$ max. For a given value of v_t, v_{t_1} in Figure 4, the weight of the glider should be adjusted to a value W_1 so that the operating point A lies on the envelope, thus giving the maximum value of $\overline{V}$. Any other weight, W_2 or W_3, will give operating points B or C.

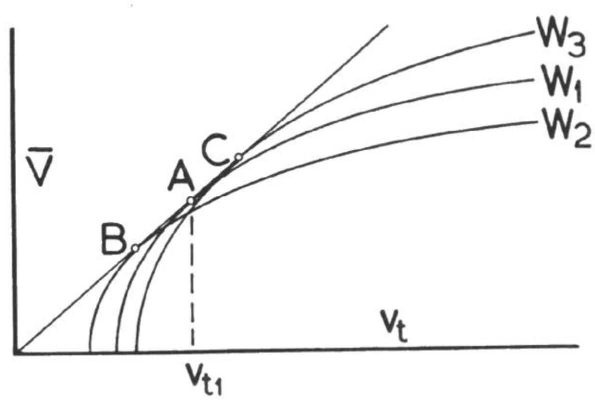

Fig. 4

Calculation for an Idealised Glider

Consider a glider whose polar is given by the simple equation (see Ref. 1)

$$Wv_s = k_1 V^3 + k_2/V \tag{5}$$

Then, by the usual analysis

$$V_o = (k_2/k_1)^{\frac{1}{4}} \tag{6}$$

and, by substituting in (5)

$$Wv_{so} = 2\,k_1^{\frac{1}{4}}\,k_2^{\frac{3}{4}}$$

At $V = rV_o$

$$Wv_s = k_1\,r^3\,V_o^3 + k_2/r\,V_o$$

and introducing (6)

$$Wv_s = (r^3 + 1/r)\,k_1^{\frac{1}{4}}\,k_2^{\frac{3}{4}}$$

Hence
$$\frac{Wv_s}{Wv_{so}} = \frac{r^4+1}{2r}$$

So that, with the above sign convention, the non-dimensional equation for the polar becomes

$$\Delta = -\frac{r^4+1}{2r} \tag{7}$$

Hence
$$S' = -\left(\frac{3r^4-1}{2r^2}\right)$$

and
$$\frac{u}{c+n} = \frac{2r^2\,(r^4-1)}{(3r^4-1)(r^4+rn-1)} \tag{8}$$

Equation (4) becomes

$$n = \frac{(2r^4-2)^2}{(r^4+1)\,2r} \tag{9}$$

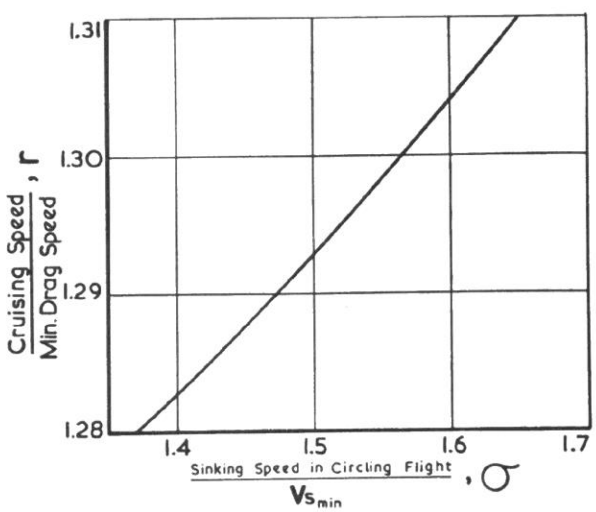

Fig. 5

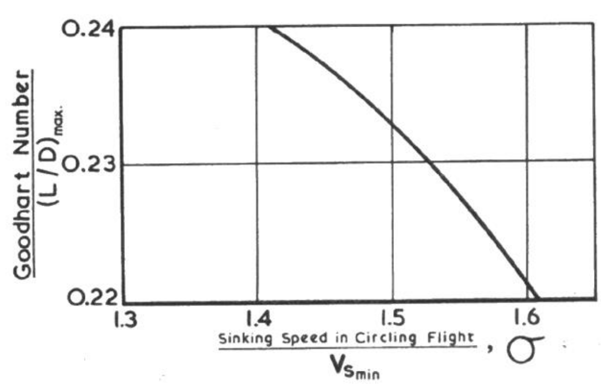

Fig. 6

Hence, given n, r may be found. In practice, the form of this equation is such that it is more convenient to choose values of r and to find the corresponding values of n.

For this idealised glider, it may be shown that

$$W v_{s\ min.} = 4 \times 3^{-\frac{3}{4}} k_1^{\frac{1}{4}} k_2^{\frac{3}{4}} \tag{10}$$

So that

$$K = \frac{v_{s\ min.}}{v_{so}} = 2 \times 3^{-\frac{3}{4}} = \frac{1}{1 \cdot 14}$$

and therefore $\sigma = n/K = 1 \cdot 14\, n$ (11)

Figure 5, showing r plotted against σ, may therefore be obtained from equations (9) and (11). For each value of δ, $\frac{u}{c+n}$ may be obtained from the corresponding values of n and r (Equation 8), and may also be plotted as shown in Figure 6.

It now remains to select a suitable value for σ. Goodhart has suggested 1.5, which, as is shown below, would seem to be reasonable. From Figures 5 and 6, if $\sigma = 1.5$, then $r =$ 1.293 and $\frac{u}{c+n} = 0.232$.

Hence for an idealised glider, and assuming that the rate of sink in circling flight is 1.5 × minimum rate of sink in straight flight, the Goodhart Number, G, is given by

$$G = 0.232 \times (L/D)\ \text{max.}$$

For a glider whose (L/D) max. is 32, the Goodhart Number would be 7.43, for example.

Justification for choosing $\sigma = 1.5$

If a real glider is considered, operating under practical conditions, it is clear that in general σ will vary from one thermal to another and even from time to time in the same thermal, since σ a function of forward speed and angle of bank. However, in order to compare different gliders on the same basis, a standard value must be chosen.

If a glider in circling flight is always flown at the speed appropriate to minimum rate of sink (which will itself depend on the angle of bank, since the corresponding lift coefficient remains constant), then the minimum rate of sink will be proportional to $(\sec\Phi)^{\frac{3}{2}}$ where Φ is the angle of bank.

i. e.
$$\sigma = (\sec\Phi)^{\frac{3}{2}} \tag{12}$$

If Φ is taken as 30°, representing a fairly typical condition, then $\sigma = 1.24$ on this theoretical basis. In practice, there is a slight increase in both induced and profile drags due to the non-uniform spanwise distribution of local airspeed and direction and to the aileron deflection. A further increment occurs due to inevitable errors and corrections in piloting. It would therefore seem reasonable to take $\sigma = 1.5$ as representing what is likely to happen in practice. It is worth noting that if $\sigma = 1.5$ in Equation (12), Φ is almost exactly 40°.

The Goodhart Number for some representative gliders.

Taking $\sigma = 1.5$, the Goodhart Number has been obtained fore some representative gliders by graphical construction. The results are given in table 1.

Type	Max. $\left(\frac{L}{D}\right)$	G	$G/\left(\frac{L}{D}\right)$max.
Weihe	29.2	6.30	0.216
Sky	27.5	6.32	0.230
Skylark 3	32	7.15	0.224
Breguet 901	36	8.05	0.224

Table 1.

Comparing the figures in the final column with the "idealised glider" value of 0.232, it will be seen that the lowest shows a deficit of only 8%. To a first order, therefore, the best lift/drag ratio can be taken as a measure of the potential performance.

Some Practical Considerations

As mentioned above, the weight of the glider on a particular day should be adjusted so that the operating point lies on the envelope of the average speed curves (e. g. at A in Fig. 4).

However, the variations in weight which would be needed to satisfy this requirement are quite excessive. Thus, if the point A normally corresponded to a thermal strength of 1.7 metres/sec, an increase in thermal strength to 2.4 metres/sec, would theoretically require the weight to be doubled. In practice, the loss in performance due to flying at the incorrect weight is very small: the operating point of the Breguet 901 with water ballast (390 kg all-up weight) corresponds to thermals of 1.89 metres/sec, but if it is flown in the same thermals without ballast (315 kg all-up weight), the loss in cross-country speed is of the order of 1 k. p. h. It is interesting to note the thermal strengths corresponding to the ideal operating points of the gliders previously considered, as shown in table 2.

Type	All-up weight		Best thermal strength	
	Pounds	kg	m/sec.	Ft/min.
Weihe	735	328	1.71	336
Sky	800	375	2.10	413
Skylark 3	790	353	1.89	372
Breguet 901	705	315	1.70	335
Breguet 901	875	390	1.89	372

Table 2.

Conclusions

1. The Goodhart Number is a non-dimensional quantity expressing the potential cross-country performance of a glider.
2. It depends on the shape of the performance curve and the maximum L/D.
3. In assessing the Goodhart Number for a given glider it is reasonable to take the minimum sinking speed in circling flight as 1.5 times that in straight flight.
4. To achieve the best performance on a given day, as described by the Goodhart Number, the weight of the glider should be suitably adjusted. However, the penalty involved in operating at a fixed weight is generally very small.

Reference

Welch and Irving, "The Soaring Pilot". John Murray, 1955, Chapter III.